MA 214 002 Calculus IV (Spring 2016)

Solutions to Quiz 7

1. Consider the non-homogeneous equation

$$y'' - 3y' + 2y = 3e^{2t}.$$

   (a) (10%) Write down the characteristic equation for the homogeneous equation $y'' - 3y' + 2y = 0$ and find its roots.

   (b) (20%) If the method of undetermined coefficients is to be used to find a particular solution $Y(t)$ of the given equation, give the *form* of $Y(t)$ that should be used.

   (c) (20%) Use the method of undetermined coefficients to find a particular solution of the given equation.

Solution: (a) The characteristic equation of the homogeneous equation $y'' - 3y' + 2y = 0$ is $r^2 - 3r + 2 = 0$, the roots of which are $r_1 = 1$, $r_2 = 2$.

(b) The right-hand side of the given non-homogeneous equation is $g(t) = 3e^{2t} = P(t)e^{st}$, where the polynomial $P(t) = 3$ and $s = 2$. Since $P(t)$ is of degree 0 and $s = 2$ appears once as a root of the characteristic equation, in the formula $Y(t) = t^j Q(t)e^{st}$ we have $j = 1$ and $Q(t) = A$, where A is a constant. Therefore the form of $Y(t)$ that should be used in the method of undetermined coefficients is:

$$Y(t) = Ate^{2t},$$

where A is a coefficient to be determined.

(c) For $Y(t) = Ate^{2t}$, we have $Y'(t) = Ae^{2t} + 2Ate^{2t}$, $Y''(t) = 4Ae^{2t} + 4Ate^{2t}$. Substituting $y = Y(t)$ into the given non-homogeneous equation, we obtain

$$Y'' - 3Y' + 2Y = A(4e^{2t} + 4te^{2t} - 3e^{2t} - 6te^{2t} + 2te^{2t}) = Ae^{2t} = 3e^{2t}.$$

Comparing coefficients, we conclude that

$$A = 3.$$

Therefore a particular solution of the given non-homogeneous equation is

$$Y(t) = 3te^{2t}.$$

2. (a) (15%) Verify that $y_1 = t^2$ and $y_2 = t^{-1}$ form a fundamental set of solutions of the homogeneous equation

$$t^2 y'' - 2y = 0, \qquad (t > 0)$$

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(b) (30%) Use the method of variation of parameters to find a particular solution $Y(t)$ of the non-homogeneous equation

$$t^2 y'' - 2y = 2t^3, \qquad (t > 0).$$

No credit will be given unless you use the method of variation of parameters.

(c) (5%) Write down the general solution of the non-homogeneous equation given in (b).

Solution: (a) Let $L(y) = t^2 y'' - 2y$. Since

$$L(y_1) = L(t^2) = t^2 \cdot 2 - 2t^2 = 0,$$
$$L(y_2) = L(t^{-1}) = t^2 \cdot 2t^{-3} - 2t^{-1} = 2t^{-1} - 2t^{-1} = 0$$

for each $t > 0$, y_1 and y_2 are solutions of the homogeneous equation in question. Since the Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 < 0 \qquad \text{for each } t > 0,$$

y_1 and y_2 constitute a fundamental set of solutions of the homogeneous equation $L(y) = 0$.

(b) In standard form, the given non-homogeneous equation reads:

$$y'' - \frac{2}{t^2} y = 2t, \qquad \text{where } t > 0.$$

Note that $g(t) = 2t$. Using the method of variation of parameters, we seek a particular solution of the form $Y = u_1 y_1 + u_2 y_2$, where u_1' and u_2' are given by

$$u_1' = \frac{1}{W} \begin{vmatrix} 0 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = \frac{-2}{-3} = \frac{2}{3}, \qquad u_2' = \frac{1}{W} \begin{vmatrix} t^2 & 0 \\ 2t & 2t \end{vmatrix} = -\frac{2}{3} t^3,$$

respectively. We take

$$u_1 = \frac{2}{3} t, \qquad u_2 = -\frac{1}{6} t^4,$$

which deliver the particular solution

$$Y = u_1 y_1 + u_2 y_2 = \frac{2}{3} t \cdot t^2 - \frac{1}{6} t^4 \cdot \frac{1}{t} = \frac{1}{2} t^3.$$

(c) The general solution of the given non-homogeneous equation can be written as

$$y = c_1 t^2 + c_2 t^{-1} + \frac{1}{2} t^3,$$

where c_1 and c_2 are arbitrary constants.