The Blind Bartender's Problem*

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Abstract

We present a two-player game with restricted information for one of the players. The game takes place on a transitive group action. The winning strategies depend on chains of structures in the group action. We also study a modification of the game with further restrictions on one of the players.

1 Introduction

Consider the following game for two players, where one player is called the Blind Bartender and the other the Antagonist. There is a square tray with four glasses, one standing in each corner of the tray. A glass can be either up (upright) or down (upside down). The Bartender is facing an edge of the tray. Thus, the four glasses occupy four positions: left-front, left-back, right-front, and right-back. Since the Bartender is blind, he cannot see the glasses, but his goal is to turn the glasses so that they will all be up or will all be down. A round in the game goes like this: the Bartender announces one or two of the four positions. After this declaration, the Antagonist is allowed to rotate the tray through a multiple of 90 degrees. By such a rotation, the glasses occupying the four positions have been permuted. Now the Bartender is allowed to touch the glasses occupying the positions he declared. He touches them and decides how he wishes to turn them. He may leave the glasses as they are, he may turn one of them, or he may turn both of them. If, after his decision, all four glasses are up, or all four glasses are down, then the Bartender has won. If not, the game continues with another round.

One can easily see that the Bartender can win this game in five moves.

Our goal in this paper is to study a generalization of this game. The essential structure of the game described above is the set of four positions for the glasses and that the glasses can be permuted cyclically through these positions. A natural way to generalize this is to give a set S of positions and a group G acting transitively on this set. A glass is standing on each element of S. As before, it may be up or down. The Bartender chooses a subset $S' \subseteq S$. The Antagonist applies some element $g \in G$ to S. The glasses the Bartender gets to touch are those sitting atop the elements in the image of S' under the action of g. One sees that our original game corresponds to the case where S consists of four elements and G is just the group of cyclic permutations of S. (To see this correspondence explicitly, one may think of the Antagonist as returning the glasses to their original positions after the Bartender has altered their states. Because the Bartender is blind, this has no effect on the game.) For the more

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general game, one is led to ask how many hands the Bartender needs, if he is to be able to win. That is, how large must $|S'|$ be? We will completely answer this question in this paper.

The major concept used to solve this general problem is *subactions*. As before, let the group G act on the set S. A subaction is a partition of the set S into blocks, such that if we apply an element from the group G to a block, we will get another block in the same partition. The subactions of an action play the same role as subgroups of a group. The blocks of a subaction correspond to the left cosets of a subgroup. Indeed, if we let a group act on itself by left multiplication, then the subactions naturally correspond to subgroups of the group.

Let us now return to the tray with the four glasses. In the game the Bartender has two sources of information. The first is through touching the glasses and feeling if they are up or down. The second is knowing if he has or has not won after a round in the game. (We assume that the Antagonist is honest enough to tell the Bartender if he has won.) Let us further restrict the information the Bartender gets, by making him wear boxing gloves on his hands. The boxing gloves ensure that he cannot feel if a glass is up or down, but he can still turn a glass. Furthermore, in each round, he must turn all of the glasses he has chosen. Is it still possible for the Bartender to win?

With some cleverness one can see that the Bartender can win in seven moves in the game with four glasses on a tray, even though he is wearing boxing gloves on both hands. But if there were only three glasses on a triangular tray, the Antagonist could force the game to go on forever.

It is natural to ask for which transitive group actions the Bartender can win the game while wearing boxing gloves. This question has a surprisingly easy answer.

In section 2 of this paper we define The Blind Bartender's Problem (without boxing gloves) formally. Then in section 3 we introduce the mathematical tools needed to state and prove the main theorems. Section 4 is devoted to solving The Blind Bartender's Problem. First we present a winning strategy for the Bartender in the case he is able to win. After that we show a strategy for the Antagonist, when the Antagonist can force the game to continue forever. In section 5 we prove the main theorem for The Blind Bartender's Problem with boxing gloves. So far we have been assuming that the action is transitive, so in section 6 we make a brief remark about the game on a nontransitive action.

The inspiration to study this game came from Martin Gardner. In [1] he presents the game with four glasses standing on a tray, where a glass can be either up or down. He asks if a two-handed Bartender can get all glasses standing up or all glasses down. The next month in [2] he shows that the Bartender can win in five moves. He also mentioned the problem in [3].

Ronald L. Graham and Persi Diaconis studied the game with n glasses standing in a cycle on a tray. That is the game $(\mathbf{Z}_n, \mathbf{Z}_n)$ in our notation, where the group acts on itself by left actions. They proved that if n is a composite integer then a $(n-2)$-handed Bartender can win the game, but if n is a prime number then he will lose.

After we had completed our analysis of the general game, it was pointed out to us that William T. Laaser and Lyle Ramshaw in [4], and Ted Lewis and Stephen Williard in [5] solve completely the question about how many hands a Bartender needs to win the game with n glasses standing cyclically on a tray (the $(\mathbf{Z}_n, \mathbf{Z}_n)$ game).

Finally, we have not seen the version of The Blind Bartender's Problem where the Bartender wears boxing gloves in the literature.

2 The Game

Let $S = \{s_1, s_2, \ldots, s_n\}$ and Σ_n denote the group of permutations of $\{1, \ldots, n\}$. For $\sigma \in \Sigma_n$, Σ_n acts on S by $\sigma(s_i) = s_{\sigma(i)}$. Let $G \subseteq \Sigma_n$ be a subgroup which acts transitively on S. That is, for every $s_i, s_j \in S$, there exists $\sigma \in G$ such that $s_{\sigma(i)} = s_j$.

Consider the following two person game, played by an Antagonist (**A**) and a Bartender (**B**).

To each $s_i \in S$ there corresponds a state $x_i \in \{-1, +1\}$, and $k \leq n$ is fixed. We say **B** has won if all x_i are the same. Play proceeds as follows.

1. **A** chooses an initial set of x_i.
2. **B** chooses an ordered m-tuple $\boldsymbol{s} = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$, $m \leq k$, such that all the i_j are distinct. **B** sends $\boldsymbol{s}$ to A.
3. **A** chooses some $\sigma \in G$ and sends the k-tuple $\boldsymbol{x} = (x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_m)})$ to **B**. Hence **B** gets to know the values in $\boldsymbol{x}$.
4. **B** is allowed to change any or all of the $x_{\sigma(i_j)}$'s in $\boldsymbol{x}$. He receives no information about any x_i not in $\boldsymbol{x}$; he is blind.
5. If **B** has not won, repeat steps 2 – 4 above.

The natural question to ask about this game is the following:

Question 1 *For what values of k can* **B** *always win?*

This paper will answer completely the above question.

3 Definitions

Given S, G, and k as in the previous section, let (S, G, k) denote the game described. When the value for k is unambiguous, we will often refer to the (S, G) game.

It is not necessary to restrict our attention to subgroups of Σ_n. Suppose G is any group acting transitively on S. Let $H \subseteq G$ be the maximal subgroup fixing every s_i. Then H is a normal subgroup and G/H is isomorphic to some subgroup of Σ_n. Hence, we may speak of the general (S, G, k) game, where G is any group acting transitively on S.

Let $\Pi[S]$ be the set of all partitions of S. Endow S with a partial ordering such that for $\pi_1, \pi_2 \in \Pi[S]$, $\pi_1 \leq \pi_2$ if and only if π_1 is a refinement of π_2. Let $\hat{0} = \{\{s\} : s \in S\}$ and $\hat{1} = \{S\}$. The partition $\hat{0}$ is the smallest element in the partial order on $\Pi[S]$ and the partition $\hat{1}$ is the largest element.

Definition 1 *A partition $\pi \in \Pi[S]$ is a* subaction *if for every $B \in \pi$ and every $\sigma \in G$, $\sigma B \in \pi$.*

Observe that if π is a subaction, G acts on π, and since G acts transitively on S, it acts transitively on π. Moreover, all the blocks of π have the same cardinality. Trivially $\hat{0}$ and $\hat{1}$ are subactions.

Let $B \in \pi$, where π is a subaction on S. Let H be the subgroup which maps the block B to itself. That is

$$H = \{h \in G \mid hB = B\} .$$

Then we may speak of the (B, H, k') game. Note that H acts transitively on B.

Definition 2 *A* saturated chain *of subactions is a chain*

$$\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

where each π_i is a subaction and there exists no subaction π such that $\pi_{i-1} < \pi < \pi_i$ for $0 < i \leq m$. We say that a saturated chain is greedy *if for all i, and for all subactions τ*

$$\tau < \pi_i \implies |\tau| \geq |\pi_{i-1}|.$$

For a fixed chain of subactions (not necessarily saturated)

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

let

$$c(S, G, \boldsymbol{\pi}) = \max_i \frac{|\pi_{i-1}|}{|\pi_i|}.$$

Let

$$c(S, G) = \min_{\boldsymbol{\pi}} \max_i \frac{|\pi_{i-1}|}{|\pi_i|} = \min_{\boldsymbol{\pi}} c(S, G, \boldsymbol{\pi}) \tag{1}$$

where both minimums are taken over all chains of subactions $\boldsymbol{\pi}$.

In each round, we refer to **B**'s choice of $\mathbf{s}$ and his subsequent change of the states of $\mathbf{x}$ as a *move*. We are interested in determining the minimal value of k such that B has a strategy to win in a bounded number of moves. Let $|S| - j(S, G)$ be the minimal k. We prove the following:

Theorem 1 *For any set S and group G acting transitively on S*

$$j(S, G) = \frac{|S|}{c(S, G)}.$$

4 Proof of Theorem 1

The proof of the theorem is divided into two sections. The first is concerned with showing that $j(S, G) \leq |S| \cdot (c(S, G))^{-1}$. In the second section, it is shown that $j(S, G) \geq |S| \cdot (c(S, G))^{-1}$.

4.1 Upper bound for $j(S, G)$

We split the problem into two cases, one where $c(S, G) > 2$ and one where $c(S, G) = 2$. The second case requires a little more work than the first. First we need the following:

Lemma 1 *Let S be a set, G be a group acting transitively on S, and*

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

be a fixed chain of subactions. If $k \geq |S| \cdot \left(1 - \frac{1}{c(S,G,\boldsymbol{\pi})}\right)$ then there is a sequence of $m = length(\boldsymbol{\pi})$ moves such that either **B** *wins in the course of making these moves, or the state of the x_i's after the m^{th} move consists of one -1 with all the others $+1$. Moreover each move entails possibly changing a -1 to $+1$ but not otherwise.*

Proof: The proof is by induction. The lemma is clearly true if $m = 1$, and so the induction base is done.

The induction step is the following. Suppose the lemma is true for all S, G, and $\boldsymbol{\pi}$ with $length(\boldsymbol{\pi}) < m$. Fix $B \in \pi_{m-1}$. By assumption the lemma is true for B, H – the maximal subgroup mapping B to B, and

$$\boldsymbol{\pi}' : \hat{0}' = \pi'_0 < \pi'_1 < \cdots < \pi'_{m-1} = B,$$

where $\boldsymbol{\pi}'$ is the restriction of $\boldsymbol{\pi}$ to B. (Obviously, $\boldsymbol{\pi}'$ is a chain of subactions for H acting on B.) Suppose the blocks of π_{m-1} are $B_1, B_2, \ldots, B_a$ where $n = |S| = ab$. Assume furthermore that $B_i = \{s_{b(i-1)+1}, \ldots, s_{bi}\}$. Let $\sigma_2, \sigma_3, \ldots, \sigma_a \in G$ be such that $\sigma_i B_1 = B_i$. Proceed as follows:

1. Choose the k-tuple $\boldsymbol{s} = (s_1, s_2, \ldots, s_{n-b}, \ldots, s_k)$. Note that $n - k \leq b$.

2. **B** receives $\boldsymbol{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n-b)}, \ldots, x_{\sigma(k)})$. **B** changes everything to $+1$. By choice of $\boldsymbol{s}$, he has changed the states of all elements in $b-1$ of the blocks to $+1$. If did not win, proceed as follows.

3. Observe $b - \frac{b}{c(S,G,\boldsymbol{\pi})} \geq b - \frac{b}{c(B,H,\boldsymbol{\pi}')}$ and that $b - \frac{b}{c(B,H,\boldsymbol{\pi}')}$ is an integer. Thus $k' = \lfloor b - \frac{b}{c(S,G,\boldsymbol{\pi})} \rfloor \geq b - \frac{b}{c(B,H,\boldsymbol{\pi}')}$. **B** chooses $\{s_{i_1}, s_{i_2}, ..., s_{i_{k'}}\} \subseteq B_1$ as the sequence of $m-1$ moves for (B_1, H, k') would dictate. Choose

   $$\boldsymbol{s} = (s_{i_1}, s_{i_2}, \ldots, s_{i_{k'}}, \sigma_2(s_{i_1}), \ldots, \sigma_2(s_{i_{k'}}), \ldots, \sigma_b(s_{i_1}), \ldots \sigma_b(s_{i_{k'}}), s'_1, \ldots, s'_{k-bk'})$$

   where $s'_1, \ldots, s'_{k-bk'}$ are arbitrary elements of B_b.

4. **B** changes everything in each consecutive string of k' $x'_i s$ as in the strategy for $(B_1, H, \boldsymbol{\pi}')$. The remaining $n - bk'$ x_i's are ignored. By the choice of $\boldsymbol{s}$ we are playing the right strategy in every B_i to achieve the desired position. If have not won continue as in (3) for at least $m-2$ times.

By hypothesis, if **B** has not won before completing the m moves, then after them, **B** knows that the states associated to the elements of each B_i include at most one -1. However, since the first move entailed turning the states of $b-1$ of the $B'_i s$ to $+1$ and no $+1$ changes into -1, **B** knows that there is exactly one -1 and $(n-1)$ number of $+1$'s. □

Lemma 2 *Let S be a set, G be a group acting transitively on S, and*

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

be a fixed chain of subactions. If $c(S,G) > 2$, then

$$j(S,G) \geq \frac{|S|}{c(S,G,\boldsymbol{\pi})}.$$

Proof: The proof is by induction on m. Let $k = |S|\left(1 - \frac{1}{c(S,G,\boldsymbol{\pi})}\right)$ and $n = |S| = ab$ where $a = |\pi_{m-1}|$.

If $m = 1$ then $c(S,G,\boldsymbol{\pi}) = |S|$. Clearly, $(S, G, |S|-1)$ can be won in no more than two moves. On the first move **B** changes $\boldsymbol{x}$ to all $+1$'s. If he did not win, then he knows that the remaining $x_i = -1$. If on his next move the $\boldsymbol{x}$ contains a -1, then he changes that x_i to $+1$ and wins. If $\boldsymbol{x}$ does not contain a -1 then he changes the entire vector to have all entries equal to -1, thereby winning. Thus, the lemma is true for all S, G, and $\boldsymbol{\pi}$ with $m = 1$. Hence the induction base is proven.

The induction argument is as follows. Assume the lemma is true for all S, G, and $\boldsymbol{\pi}$ with length less than m. By assumption, the lemma is true for $B \in \pi_{m-1}$, H - the maximal subgroup mapping B to B, and

$$\boldsymbol{\pi}' : \hat{0}' = \pi'_0 < \pi'_1 < \cdots < \pi'_{m-1} = B,$$

where $\boldsymbol{\pi}'$ is the restriction of $\boldsymbol{\pi}$ to B.

Since $b - \frac{b}{c(B,H,\boldsymbol{\pi}')}$ is an integer we have the inequality $\lfloor b - \frac{b}{c(S,G,\boldsymbol{\pi})} \rfloor \geq b - \frac{b}{c(B,H,\boldsymbol{\pi}')}$. Let $k' = \lfloor b - \frac{b}{c(S,G,\boldsymbol{\pi})} \rfloor$. We describe an explicit strategy for **B** to win (S, G, k).

1. Proceed as in Lemma 1 to arrive at the position where at most one x_i is -1, and all the others are $+1$. If **B** has not won, then there is exactly one $x_i = -1$ after his first m moves.

2. Without loss of generality, we may assume the blocks of π_{m-1} are

$$B_1 = \{s_1, s_2, \ldots, s_b\}, \ldots, B_a = \{s_{n-b+1}, s_{n-b+2}, \ldots, s_n\}$$

and the corresponding states are

$$\boldsymbol{x}_1 = \{x_1, x_2, \ldots, x_b\}, \ldots, \boldsymbol{x}_a = \{x_{n-b+1}, x_{n-b+2}, \ldots, x_n\}.$$

Also, we may assume that $x_1 = -1$ and $x_i = +1$ for $i > 1$. Define two variables:

$$w = (\# \text{ of } +1 \text{ elements in } \boldsymbol{x}_1) = b - 1$$

and

$$z = (\text{state of } S \backslash B_1) = 1.$$

Also, choose $\sigma_2, \ldots, \sigma_a \in G$ such that $\sigma_i B_1 = B_i$. Keep these fixed for the whole game.

3. Choose the k-tuple $\boldsymbol{s} = (s_1, s_2, \ldots, s_{n-b}, \ldots, s_k)$. This is possible since $n - k \leq b$.

4. **B** receives $\boldsymbol{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n-b)}, \ldots, x_{\sigma(k)})$. If **B** receives the one -1, he changes it to $+1$ and wins. If not, **B** changes $x_{\sigma(i)}$ to -1 for $i \leq n - b$, and leaves the rest the same. Also change the variable z to -1.

Remark: The idea behind this is that if **B** can distinguish which x_i's come from $\boldsymbol{x}_1$ in each $\boldsymbol{x}$ he receives, he may then play the strategy for (B_1, H, k') on $\boldsymbol{x}_i$. Now, if **B** chooses at least $k' > b/2$ elements from each block, and

- if $w \geq b/2$ and $z = -1$, **B** can distinguish which x_i in $\boldsymbol{x}$ come from $\boldsymbol{x}_1$ and which do not, for if **B** chooses $s_{i_1}, \ldots, s_{i_{k'}}$ from the same block, then the $x_{\sigma(s_i)}$ will belong to the same block. If the $x_{\sigma(s_i)}$'s are all -1, then since $w \geq b/2$, they must all belong to some $\boldsymbol{x}_j \neq \boldsymbol{x}_1$. Also, if they are not all -1, then they must belong to $\boldsymbol{x}_1$.
- Similarly if $w < b/2$ and $z = +1$.

Since we assumed $c(S, G) > 2$ this is possible, for $k' \geq b - \frac{b}{3} > \frac{b}{2}$.

5. Choose $\{s_{i_1}, s_{i_2}, \ldots, s_{i_{k'}}\} \subseteq B_1$ as the strategy for (B_1, H, k') dictates. Let

$$\boldsymbol{s} = (s_{i_1}, s_{i_2}, \ldots, s_{i_{k'}}, \sigma_2(s_{i_1}), \ldots, \sigma_2(s_{i_{k'}}), \ldots, \sigma_b(s_{i_1}), \ldots \sigma_b(s_{i_{k'}}), s'_1, \ldots, s_{n-bk'})$$

where $s'_1, \ldots, s_{n-bk'}$ are arbitrary elements of B_b.

6. By the remark above, **B** can distinguish which x_i came for $\boldsymbol{x}_1$ in the $\boldsymbol{x}$ he receives. Suppose **A** applied $\sigma \in G$, so that $\sigma B_1 = B_j$. By the choice of the k-tuple $\boldsymbol{s}$, it is as though **B** were playing the (B_1, H, k') game and **A** had applied $\sigma\sigma_j$. **B** can make the changes dictated by the strategy for (B_1, H, k'). The $k - bk'$ elements at the end of $\boldsymbol{x}$ are ignored.

7. **B** knows exactly how many $+1$'s he changed to -1 in $\boldsymbol{x}_1$, so he recalculates w.

8. If $w = 0$ or $w = a$, **B** can win the game. He merely chooses $\boldsymbol{s}$ as in (3). If the $\boldsymbol{x}$ he receives contains all of $\boldsymbol{x}_1$ (since he did not win, the other blocks must be in the opposite state, so he can distinguish them from B_1), then **B** changes it and wins. If $\boldsymbol{x}$ does not contain all of $\boldsymbol{x}_1$ then by the choice of $\boldsymbol{s}$ it must contain all of the other blocks. **B** changes them to agree with $\boldsymbol{x}_1$ and wins.

9. If $0 < w < b/2$ then **B** makes sure that $z = +1$. If $z = -1$ he must change the other blocks. This is done in much the same way as he did it in (3) and (4). As in (8), **B** either receives all of $\boldsymbol{x}_1$ or all of the other $\boldsymbol{x}_i$. If a $+1$ appears in the first $n - b$ entries of $\boldsymbol{x}$, then **B** has received all of $\boldsymbol{x}_1$ and he knows which string of length b it is, so he may change it and win. If a $+1$ does not appear in the first $n - b$ entries of $\boldsymbol{x}$, then those entries must correspond to $\boldsymbol{x}_2, \ldots, \boldsymbol{x}_a$. **B** changes all these entries to $+1$. Thus he has changed z to $+1$, as desired. He begins again with (5).

By hypothesis, (B_1, H, k') can be won in a bounded number of moves. Since the strategy for (S, G, k) outlined above involves playing a bounded number of moves for each move in the strategy for (B_1, H, k'), the position corresponding to (8) is reached in a bounded number of moves. Hence, (S, G, k) can be won in a bounded number of moves. Thus, the lemma is true for S, G, and $\boldsymbol{\pi}$. □

Lemma 3 *Let S be a set, and G be a group acting on S such that $c(S, G) = 2$. Suppose*

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1}$$

is a chain such that $c(s, G, \boldsymbol{\pi}) = 2$. Then

$$j(S, G) \geq \frac{|S|}{2}.$$

Moreover, a strategy exists such that for each recalculation of w after having reached the position described by Lemma 1, $w = b/2$ only if every 2-block in π_1 has one element with the state $+1$ and the other element with the state -1 or the game is equivalent to the $(S/\sim, G, \lfloor k/2 \rfloor)$ game, where $s \sim r$ if s, r belong to the same 2-block. Where w is defined as in the proof of Lemma 2.

Proof: Obviously, $|S| = 2^m$. The proof is by induction on m. The lemma is clearly true for $m = 1$, thus the induction base is done.

Assume the lemma is true for all S, G, such that $|S| = 2^\alpha$, where $\alpha < m$. In particular the lemma is true for $B \in \pi_{m-1}$ and H the maximal subgroup mapping B to B.

Let $B = B_1$, B_2 be the blocks of π_{m-1}. The strategy follows that outlined in the proof of lemma 2 with one difference. The only situation where we would have trouble distinguishing $\mathbf{x_1}$ and $\mathbf{x_2}$, is when $k = |S|/2$, $k' = b/2$, $w = b/2$, and $\boldsymbol{x}$ consists of all -1's or all $+1$'s. However, if this occurs, **B** may do the following.

1. Suppose $\boldsymbol{x} = (x_{i_1}, ..., x_{i_{k'}}, x_{j_1}, ..., x_{j_{k'}})$. If any two of the x_i's come from the same block (something **B** would know), then by the induction hypothesis, the game is equivalent to $(S/\sim, G, k/2)$. If not, **B** chooses either $l \in \{i, j\}$. **B** changes $x_{l_1}, ..., x_{l_{k'}}$ all to the state opposite that which he received them in. By the assumption on (B_1, H, k'), **B** knows that either one block is entirely in $+1$ state and the other is in the -1 state, or every 2-block has one element in $+1$ state and the other in the -1 state.

2. Choose $\boldsymbol{s}$ to be a whole block. If get $\boldsymbol{x}$ all the same, then change every element and **B** wins. Otherwise, change nothing.

3. Choose $\boldsymbol{s}$ to contain one element from every 2-block of π_1. Change the state of every entry in $\boldsymbol{x}$. Now both elements of each 2-block are in the same state. Define an equivalence relationship on S, where $s \sim r$ if s, r belong to the same 2-block. The game is now equivalent to the $(S/\sim, G, k/2)$ game. By hypothesis, this game is solvable in a bounded number of moves. Hence, the lemma is true.

□

4.2 Lower bound for $j(S, G)$

Lemma 4 *Let S be a set and G be a group acting transitively on S. Suppose*

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

is a greedy chain of subactions. Then

$$j(S, G) \leq \frac{|S|}{c(S, G, \boldsymbol{\pi})}.$$

Proof: We have that

$$c = c(S, G, \boldsymbol{\pi}) = \max_i \frac{|\pi_{i-1}|}{|\pi_i|}.$$

For some i this maximum is achieved. Say that for j we have that

$$c = \frac{|\pi_{j-1}|}{|\pi_j|}.$$

We modify the notation as follows: for $s \in S$, let x_s denote the corresponding state. Call a position *losing* if there exists a block $B \in \pi_j$ and two elements $s, t \in B$ such that $x_s \neq x_t$.
Claim: If

$$k < |S| \left(1 - \frac{1}{c}\right)$$

then **A** can guarantee that from a losing position the game will always go to another losing position.

This claim implies that with $k < |S| \left(1 - \frac{1}{c}\right)$ **A** can guarantee that there will always be two x_i's that are different. Hence **A** can continue the game forever.

Proof of claim: Let $H \subseteq S$ be the set of members of the k-tuple that **B** chooses. Then we have

$$|H| < |S| \left(1 - \frac{1}{c}\right)$$

Let $U = S - H$, then U is the set of elements that **B** does not get any information about. Hence

$$|U| > |S| \frac{1}{c} = |S| \frac{|\pi_j|}{|\pi_{j-1}|}.$$

There are $|\pi_j|$ blocks in π_j. By the Pigeon Hole Principle, there is a block $C \in \pi_j$ such that

$$|T| = |U \cap C| > |S| \frac{1}{|\pi_{j-1}|},$$

where $T = U \cap C$.

Define a relation $\sim'$ on S by $s \sim' t$ if there exist $g \in G$ such that $gs, gt \in T$. Let $\sim$ be the transitive closure of the relation $\sim'$. That is, $s \sim t$ if and only if there exists a sequence of elements $s = u_0, u_1, \ldots, u_e = t$ and a sequence of group elements $g_1, \ldots, g_e \in G$ such that $g_i u_{i-1}, g_i u_i \in T$ for all $i = 1, \ldots, e$. The equivalence classes of $\sim$ form the blocks of a partition. Call this partition σ. From the above formula it is easy to see that σ is a subaction.

Notice also that $s \sim' t$ implies that s and t lie in the same block of π_j. Hence when we take the transitive closure we see that if $s \sim t$ then s and t lie in the same block of π_j. We conclude that $\sigma \leq \pi_j$.

Notice that an element $s \in T$ is related to at least $|T| - 1$ other elements by the relation $\sim'$. Hence s is related to at least $|T| - 1$ other elements by the relation $\sim$. Hence the size of an equivalence class of $\sim$ is greater than equal to $|T|$. That is for $D \in \sigma$ we have that $|D| \geq |T|$. We can write this as

$$\frac{|S|}{|\sigma|} \geq |T|.$$

Now putting our two inequalities together we have that

$$\frac{|S|}{|\sigma|} \geq |T| > |S| \frac{1}{|\pi_{j-1}|}.$$

Hence

$$|\sigma| < |\pi_{j-1}|$$

But we know that $\boldsymbol{\pi}$ is a greedy chain of subactions. From the fact that $\sigma \leq \pi_j$ we conclude that $\sigma = \pi_j$.

But the position we are playing on is a losing position. Hence we have s and t in the same block of π_j such that $x_s \neq x_t$. But s and t lie in the same block of π_j, thus $s \sim t$. Therefore, we have that $s = u_0, u_1, \ldots, u_e = t$ and $g_i \in G$ so that $g_i u_{i-1}, g_i u_i \in T$. But $u_0 \neq u_e$, hence there must an i such that $x_{u_{i-1}} \neq x_{u_i}$. Notice that u_{i-1} and u_i lie in the same block of π_j. Moreover $gu_{i-1}, gu_i \in T \subseteq U$. Let **A** apply g_i to the position of the game. Then $x_{gu_{i-1}}$ and x_{gu_i} will not be in the k-tuple that **A** sends to **B**. Hence **B** will not be able to change the values of $x_{gu_{i-1}}$ and x_{gu_i}, and therefore cannot win. Moreover, **B** has to leave a losing position after his move.

Hence the proof of the lemma is done. $\square$

4.3 Proof of Theorem 1

Proof: By lemma 2 and lemma 3 we have that

$$j(S,G) \geq \frac{|S|}{c(S,G,\boldsymbol{\pi})}.$$

Taking the maximum over all chains $\boldsymbol{\pi}$ gives

$$j(S,G) \geq \frac{|S|}{c(S,G)}.$$

Observe that we can find a greedy chain $\boldsymbol{\pi}'$ by first choosing the maximal subaction $\hat{1}$, and then recursively choosing π_{i-1} as the subaction with the smallest number of blocks such that $\pi_{i-1} < \pi_i$. For a greedy chain $\boldsymbol{\pi}'$ we know from lemma 4 that

$$j(S,G) \leq \frac{|S|}{c(S,G,\boldsymbol{\pi}')}.$$

Hence we conclude that

$$j(S,G) = \frac{|S|}{c(S,G)}.$$

□

Corollary 1 *The minimum $c(S,G) = \min_{\boldsymbol{\pi}} c(S,G,\boldsymbol{\pi})$ is achieved by all greedy chains $\boldsymbol{\pi}$.*

Corollary 2 *If G is a solvable group acting on itself by left actions, that is, the game is (G,G). Then*

$$j(S,G) = \frac{|S|}{p(G)}$$

where $p(G)$ is the largest prime number dividing the order of G.

Corollary 3

$$j(\mathbf{Z}_n, \mathbf{Z}_n) = \frac{|n|}{p(n)}$$

where $p(n)$ is the largest prime number dividing n.

One generalization of the game is to let the glasses be in more states than two. For example a glass can be up, down or sideways. Let the set of states that a glass can be in be $X = \{x_1, x_2, \ldots, x_q\}$.

Will this change in the rules make any substantial changes to the game? No, the Bartender will only win if he has at least $|S|\left(1 - \frac{1}{c(S,G)}\right)$ hands. By Lemma 1 he can reduce the game to where the glasses are in two different states, x_1 and x_i, even though he does not know which state x_i is. He now plays the game assuming that the two states are x_1 and x_2. If he does not win, he continues to play the game assuming that the states are x_1 and x_3. At some point he is playing with the right assumption, and thus he has reduced the situation to a game with only two states, where he knows how to win.

If the Bartender has less than $|S|\left(1 - \frac{1}{c(S,G)}\right)$ hands, it is clear that the Antagonist will still win.

5 Boxing gloves

Let G act transitively on S. Consider the boxing glove modification of the game between the Antagonist (**A**) and the Bartender (**B**).

To each $s_i \in S$ there corresponds a state $x_i \in \{-1, +1\}$. We say **B** has won if all x_i are the same. Play proceeds as follows.

1. **A** chooses an initial set of x_i.
2. **B** chooses $k \leq n$ and an ordered k-tuple $\boldsymbol{s} = (s_{i_1}, s_{i_2}, \ldots, s_{i_k})$ such that all the i_j are distinct. **B** sends $\boldsymbol{s}$ to A.
3. **A** chooses some $\sigma \in G$ and multiplies the k-tuple $\boldsymbol{x} = (x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)})$ by -1.
4. If **B** has not won, repeat steps 2 and 3 above.

Observe that in this game **B** does not get any information about the x_i's. The only piece of information he gets is if he won or not. Note that k changes in each round of the game. Moreover we can assume that $k \leq \frac{n}{2}$ in each round, since multiplying a k-tuple by -1 is equivalent to multiplying the complemented $(n-k)$-tuple by -1. The question to ask about this game is the following.

Question 2 *For which group actions (S, G) can* **B** *always win?*

Theorem 2 **B** *can win if and only if*

$$c(S, G) = 2.$$

Proof: First we prove that **B** cannot win when $c(S, G) > 2$. By corollary 1 we can take a greedy chain $\boldsymbol{\pi}$ so that

$$c(S, G) = c(S, G, \boldsymbol{\pi}).$$

By lemma 4 we know that **B** need at least $k \geq n\left(1 - \frac{1}{c(S,G)}\right) > \frac{n}{2}$. This leads to a contradiction since we could assume that $k \leq \frac{n}{2}$. Thus if $c(S, G) > 2$ then **A** can force the game to continue forever.

What remains to prove is that if $c(S, G) = 2$ then **B** is able to win. Thus we have a chain

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{m-1} < \pi_m = \hat{1},$$

so that for all i

$$\frac{|\pi_{j-1}|}{|\pi_j|} = 2.$$

Notice that $|S| = 2^m$. The proof will be by induction.

Claim: There is a sequence of $2^{2^m - 1} - 1$ moves such that **B** will win if he applies this sequence.

Proof of claim: Clearly this is true if $m = 1$, fo then the winning move is to change one of the two x_i's.

Suppose we have the strategy for $m - 1$. Notice that $|\pi_1| = 2^{m-1}$ and that G acts transitively on π_1. Hence we can play the game on π_1. That is, we are assuming that we can win the game (π_1, G).

Define the state of a block B as $x_B = \prod_{s \in B} x_s$ for $B \in \pi_1$. Observe that the states x_B do not need to agree with the state of all the members of the block B.

Let $B \in \pi_1$, then B has only two elements. Note that $x_B = 1$ if and only if the elements of B are in the same state. In such a case the block has an *underlying state*, the state of its elements.

By the induction hypotheses, there is a sequence of moves $\alpha_1, \alpha_2, \ldots, \alpha_N$ which guarantees that **B** will win in the game (π_1, G). Here $N = 2^{2^{m-1}-1} - 1$.

Define three moves in the game (S, G).

1. Let β_i be the move that changes one element in the block $B \in \pi_1$, if α_i changes the state of the block B in the game (π_1, G).

2. Let γ_i be the move that changes both elements in the block $B \in \pi_1$, if α_i changes the state of the block B in the game (π_1, G).

3. Let δ be the move that changes one element in each block $B \in \pi_1$.

Each of these moves changes at most $\frac{n}{2}$ states.

Observe that the move β_i behaves like α_i on the states of the blocks of π_1. Moreover, γ_i does not change any state of the blocks of π_1. Lastly, δ changes all the states of the blocks in π_1.

Consider the sequence of moves

$$\mu = \gamma_1, \gamma_2, \ldots, \gamma_N$$

This sequence will detect if we start in a position in the game where all the blocks were in the state $+1$. If every block is in state $+1$, every block has a underlying state. By playing the sequence μ of moves, we are playing the game (π_1, G) on the underlying states of the blocks. Notice that if α_i changes the state of a block B, then γ_i will change the underlying state of that block B. Hence the two games are equivalent. But by the induction hypothesis the sequence $\alpha_1, \ldots, \alpha_N$ is a winning strategy. That means that at some point all the blocks will be in the same state. Thus in the equivalent game all block will have the same underlying state, which implies that at some point in the (S, G) game all the elements of S are in the same state, and at that point **B** will win.

Observe that the sequence μ does not change the state x_B of a block $B \in \pi_1$.

Study now the following sequence of length $2N + 1$.

$$\nu = \mu, \delta, \mu$$

This sequence will detect if we started in a situation where all the blocks were in the same state. First it checks if all the blocks are in the state $+1$. If not, it continues by reversing the state of every block, and checks now if the states are all $+1$. That is, it checks if the states started out all being -1.

Now consider the sequence:

$$\nu, \beta_1, \nu, \beta_2, \nu, \ldots, \nu, \beta_N, \nu$$

Notice that the subsequence $\beta_1, \ldots, \beta_N$ plays the game on the blocks of π_1 and the states x_B. (We don't need to consider the δ moves inside ν. It just reverses all the states of the blocks, which does not affect the game.) At one point all the blocks will be in the same state. Then the sequences continue with a ν-sequence, that will at some point make all the states of the elements the same. Hence this is a winning strategy.

The length of this winning sequence is

$$\begin{aligned} N + (N+1)(2N+1) &= 2 \cdot (N+1)^2 - 1 \\ &= 2 \cdot \left(2^{2^{m-1}-1}\right)^2 - 1 \\ &= 2^{2^m - 1} - 1 \end{aligned}$$

which is the value we wanted. This completes the claim, and thus we have proven the theorem. $\square$

Notice that $2^{2^m-1}-1$ is the least number of moves of a winning strategy. To see this assume that **A** always choses the unit element of the group. Then there is $2^n - 2 = 2^{2^m} - 2$ possible situations. But the orientation of them will not matter, hence we only have half. We need then to be able to go through all these situations, hence we need at least $2^{2^m-1}-1$ moves.

We can also see that the maximal number of states we alter in a winning strategy needs be 2^{m-1}. Assume that $k < 2^{m-1}$. Then even if we get information about these k states, theorem 1 tells us that we cannot win.

6 Nontransitive Actions

In the previous sections, we have restricted our attention to only those groups G which act transitively on S. In this section we briefly indicate how the earlier results can be extended easily to an arbitrary group action.

Definition 3 *Let G be a group acting on the left on a set S. An orbit S' is a subset of S such that $GS' = S'$ and no nonempty subset of S' has the same property.*

Observe that the orbits of the action form a partition of the set S. Moreover, G acts transitively on each orbit of the action of the group G on the set S.

Theorem 3 *Let S be a set and G a group acting on S. Write*

$$S = \bigcup_{i=1}^{m} S_i$$

where the S_i's are disjoint orbits such that $|S_1| \geq |S_2| \geq \cdots \geq |S_m|$. Then the (S, G, k) game is winnable from any *starting position if and only if*

$$k \geq \max\left(|S_2|,\ |S_1| - j(S_1, G),\ \ldots,\ |S_m| - j(S_m, G)\right).$$

Sketch of Proof: The sufficiency follows easily from Theorem 1. **B** can play the following strategy. Since $k \geq |S_1| - j(S_1, G)$ he has a winning strategy for the (S_1, G, k) game. Since $k \geq |S_2|$, there are two sequences δ_{+1} , δ_{-1}, both containing a finite number of moves, such that δ_x changes all the states of S_i $(i \geq 2)$ to x, where $x \in \{-1, +1\}$. Between each move of the strategy for (S_1, G, k) **B** plays δ_{+1} and δ_{-1}. Clearly, this gives a winning strategy for (S, G, k).

In the other direction, it is clear again by Theorem 1 that we must have

$$k \geq \max\left(|S_1| - j(S_1, G),\ \ldots,\ |S_m| - j(S_m, G)\right).$$

Also, if (S, G) is winnable then so is $(S_1 \cup S_2, G)$, hence to show that we must have $k \geq |S_2|$, it suffices to consider the case $m = 2$.

Assume $m = 2$ and $k < \max(|S_2|,\ |S_1| - j(S_1, G),\ |S_2| - j(S_2, G))$. We may further assume that $|S_2| \geq |S_1| - j(S_1, G)$ and $|S_2| \geq |S_2| - j(S_2, G)$, for if not, the theorem follows from the above observations. Suppose a strategy exists. There are two possible situations for states before the final move.

1. Neither S_1 or S_2 have all their elements in the same state. Let

$$\boldsymbol{\pi} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_m = \hat{1}$$

$$\boldsymbol{\tau} : \hat{0} = \tau_0 < \tau_1 < \cdots < \tau_n = \hat{1}$$

   be two chains of subactions for S_1 and S_2 respectively.

   Let i and j be maximal indices such that all the blocks of π_i and τ_j have all elements in a single state. If the next move is to win, then the Bartender must be changing blocks in π_i and τ_j. Hence, by Lemma 4, he must select greater than

$$|S_1| - \frac{|S_1|}{\max_{s \geq i} \frac{|\pi_s|}{|\pi_{s-1}|}} \geq \frac{|S_1|}{2}$$

   elements from S_1 and

$$|S_2| - \frac{|S_2|}{\max_{s \geq j} \frac{|\tau_s|}{|\tau_{s-1}|}} \geq \frac{|S_2|}{2}$$

   elements from S_2. But,

$$\frac{|S_1|}{2} + \frac{|S_2|}{2} \geq |S_2|.$$

   Thus, this cannot be the condition of the states before the last move.

2. At least one of the S_i has a single state. By hypothesis, $k < |S_2| \leq |S_1|$, so not all of one S_i can be chosen by the Bartender. Suppose all of S_2 is all one state, say $+1$. Then there is at least one element of S_1 with the state -1. Since G acts transitively on S_1, the Antagonist can ensure that the Bartender is not sent that -1. Because the Bartender was unable to choose all of S_2, the resulting position is a losing one. Reversing the roles of S_1 and S_2 in the above remarks completes the case.

This concludes the sketch of the proof. □

We leave it to the reader to formulate the corresponding result for the boxing glove game with an arbitrary group action. It is similar to the above.

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