Basic intro to ∞-categories

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(Basically copied from my own note last summer, see relative full version at http://ms.uky.edu/~ali266/Research/Note%20for%20infty%20cat.pdf)

1 Definitions, Simplicial set as a model

Come to infinity category, our primary goal is to have efficient ways to encode coherence. And because ∞-category is something falls in between 1-category and groupoid, the technical tool we are using as a model to encode the structure is simplicial sets.

elaborate this

Definition 1.1. A **simplicial set** is a functor from the simplex category $\mathbf{\Delta^{op}}$ to **Set** equipped with the standard homomorphisms of presheaves.

Example 1.2. Suppose $\mathcal{C}$ is a small category. $\mathcal{C}$ can be encoded as a simplicial set: $\mathcal{N}(\mathcal{C})$= **nerve** of $\mathcal{C}$. It is defined as follows

$$\mathcal{N}(\mathcal{C})_n = \{\text{strings of n composable morphisms in } \mathcal{C}\} = \{X_0 \to X_1 \to ... \to X_n\} = \mathrm{Fun}([n], \mathcal{C}).$$

where $[n] = \{0 \to 1 \to ... \to n\}$.

$\mathcal{N}(\mathcal{C})$ remembers the category $\mathcal{C}$: the nerve functor $\mathcal{N} : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful. This motivates the definition of an ∞-category (or a quasi-category).
Denote $\Delta^n = \mathcal{N}([n])$, then the n-th level of a sset X satisfies $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X_\cdot) \cong X_n$

Let's first introduce the notion of an "∞-groupoid".

Definition 1.3. A simplicial set K is called a **Kan complex** if for all n and $0 \leq i \leq n$, any map $\Lambda^n_i \to K$ extends to a map $\Delta^n \to K$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Remark 1.4. Note that the extended maps do not need to be unique!

Example 1.5.

1. We have the functor of taking singular homology: **Top** $\xrightarrow{Sing}$ **sSet**, $Sing_\bullet X$ is a Kan complex.

2. Our nerve functor: **Cat** $\xrightarrow{\mathcal{N}}$ **sSet**, remember that $\mathcal{N}$ is fully faithful. It will take 1-groupoids to Kan complexes.

One thing also considered special about the simplicial sets coming from the nerve is they satisfy the following property of lifting:

Proposition 1.6. *A simplicial set X is coming from a category (is a nerve of some 1-category) if it has unique lifting with respect to* **inner horns**, *in other words, if for all n and $0 < i < n$, any map $\Lambda_i^n \to X$ extends uniquely to a map $\Delta^n \to X$.*

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Remark 1.7. [HTT, 1.1.2.3] The only intersections known of Kan complexes and nerve of categories are groupoids.

So now let us generalize this proposition a little bit, to give a definition of ∞-category via the model of quasicategory.

Definition 1.8. [Rezk, 6.2] An **∞-category** (quasicategory in Rezk's note) is a simplicial set X s.t. for all n and $0 < i < n$, any map $\Lambda_i^n \to X$ admits an extension to a map $\Delta^n \to X$.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Denote $\mathbf{Cat}_\infty$ to be the category of ∞-categories.

In other words, the function $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n, X)$ induced by the inclution of the horn $\Lambda^n \to \Delta^n$ is surjective. The idea is we want the weakest structure with an associated homotopy category and representatives for each morphisms in the homotopy category.

I said the word "homotopy", but how do we get the right notion of homotopy in the world of ∞-category? First we need to look at the maps between two objects. Our first reasonable thing is to define the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ as the pullback, where $\mathcal{C}_n$ is the set of n-morphisms in $\mathcal{C}$.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "d_0 \mid d_1"] \\ * \arrow[r, "(x,y)"] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

How do we make composition concretely defined on this set of morphisms? We have a candidate: If

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] & & 2 \end{tikzcd}$$

we will have σ as a wittness of homotopy to fill in the triangle:

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "f \circ g"'] & & 2 \end{tikzcd}$$

Proposition 1.9. *All candidates are homotopic.*

From this idea, we can actually define an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ called homotopy.

Definition 1.10. [Groth, defn 1.11] Two morphisms $f, g : x \to y$ in an ∞-category $\mathcal{C}$ are homotopic if there exists a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ such that $\partial\sigma = (g, f, id_x)$:

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

Such σ is called a homotopy between f and g.

Lemma 1.11. *$f \simeq g$ gives an equivalence relation on* $\mathrm{Hom}_{\mathcal{C}}(x, y)$.

Proof.

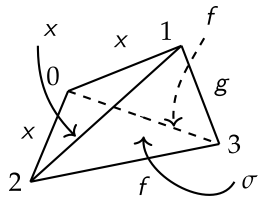
- reflection:

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

- compose: (just put together two triangles)

$$\begin{tikzcd} & x \arrow[d, "g"] \arrow[dr, "k"] & \\ x \arrow[ur, "id_x"] \arrow[r, "f"'] & y \arrow[r, "h"'] & z \end{tikzcd}$$

- symmetric:



The Λ^3_2-horn $(\sigma, id_f, -, id_x)$

Where σ is the homotopy from f to g. This horn can be filled in, for we have lifting for $\Lambda^3_2 \to \mathcal{C}$.

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Since we have the notion of homotopy, we can define the homotopy category of an ∞-category. This gives a functor from **Cat** to $\mathbf{Cat}_\infty$ as a left adjoint to the nerve functor.

$$h\colon \mathbf{Cat}_\infty \underset{\perp}{\rightleftarrows} \mathbf{Cat} : \mathcal{N}$$

Definition 1.12. The htpy cat $h\mathcal{C}$ defined as

- objects: $\mathcal{C}_0$
- morphisms: homotopy classes $\mathrm{Hom}_{\mathcal{C}}(x, y)/\sim$

is a 1-cat.

Proposition 1.13. *Let $\mathcal{C}$ be an ∞-cat and $\mathcal{D}$ be a 1-cat, then Maps from $\mathcal{C}$ to $\mathcal{D}$ factor through $h\mathcal{C}$.*

$$\begin{tikzcd} & h\mathcal{C} \arrow[dr] & \\ \mathcal{C} \arrow[ur, dashed] \arrow[rr, "f"] & & \mathcal{D} \end{tikzcd}$$

As a consequence, for a space X, the map from X to $\mathcal{D}$ factor through its fundamental groupoid $\Pi_1 X$.

Remark 1.14. On the other side, a map from $\mathcal{D}$ to $\mathcal{C}$ is a map of simplicial sets $\mathcal{N}(\mathcal{D}) \to \mathcal{C}$. So for $D_0 \to D_1 \to D_2$ in $\mathcal{D}$, we choose a 2-simplex in $\mathcal{C}$ that witnesses that these are composable.

2 Under and over categories, colimit and limit of ∞-categories

Let's start with 1-category. First we need to introduce the join of two categories. Roughly speaking the join $A \star B$ of A and B is the thing to make both $A \hookrightarrow A \star B$ and $B \hookrightarrow A \star B$ fully faithfully included. And the nerve functor is compatible with such construction, i.e. there is a natural isomorphism

$$\mathcal{N}(A) \star \mathcal{N}(B) \to \mathcal{N}(A \star B)$$

(See join for 1-cat in [HTT, section 1.2.8])

Definition 2.1. Given a category B and an object $x \in B$, the **overcategory** $B_{/x}$ has

- objects: morphisms in B target at x: $y \to x$
- morphisms: commutative triangles:

$$\begin{tikzcd} y \arrow[rr] \arrow[dr] & & z \arrow[dl] \\ & x & \end{tikzcd}$$

Similarly, the **undercategory** $B_{x/}$ has

- objects: morphisms in B coming out from x: $x \to y$
- morphisms: commutative triangles:

$$\begin{tikzcd} & x \arrow[dl] \arrow[dr] & \\ y \arrow[rr] & & z \end{tikzcd}$$

Let T denote the terminal object in **Cat**, by 24.1 in[Rezk], the over and undercategory notion can be reformulated in terms of joins, due to the observation that the objects of $B_{x/}$ correspond to the set of functors $f : [0] \star T \to B$ with $f|T = x$, and the 1-morphisms correspond to $g : [1] \star T \to B$ with $g|T = x$.

More generally, given a functor $p : A \to B$ of categories, we obtain over and undercategories $B_{p/}$ and $B_{/p}$ defined as follows. The category $B_{/p}$ has

- objects: functors $f : [0] \star A \to B$ such that $f|A = p$
- morphisms $f \to f_0$: functors $g : [1] \star A \to B$ such that $g|A = p$

Likewise, the category $B_{p/}$ is defined by switching A to the left side of the join. The slice construction and the join construction satisfies a universal property (an adjunction): For every 1-category C,

$$\mathrm{Fun}(C, B_{/p}) \cong \mathrm{Fun}_p(C \star A, B) \cong \hom_{\mathbf{Cat}_{A/}}(A \to C \star A, A \xrightarrow{p} B)$$

where the later two denote all the functors $C \star A \to B$ such that the triangle commutes:

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "p"] & \\ C \star A \arrow[rr] & & B \end{tikzcd}$$

So in ∞-cat setting, we want to have similar slice construction and universal property with respect to the join construction of simplicial sets. We start with defining the join for two simplicial sets.

Definition 2.2. Let $K, L \in \mathbf{sSet}$, the join $K \star L$ is the simplicial set such that the n-th layer is defined as

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j \ , \ n \geq 0$$

We can define the slices of ∞-categories using the join construction.

Proposition 2.3. *[Groth, Prop 2.17] Let $p : L \to \mathcal{C}$ be a map of simplicial sets and $\mathcal{C}$ an ∞-category, there is an ∞-category $\mathcal{C}_{/p}$ characterized by the following universal property: For each $K \in$* **sSet**,

$$\mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C}_{/p}) \cong \mathrm{Hom}_{\mathbf{sSet}_{L/}}(L \to K \star L, L \to \mathcal{C})$$

We call the ∞-category $\mathcal{C}_{/p}$ the **∞-category of cones** *on p.*

The above adjunction gives rise to two functors as left adjoints from **sSet** to $\mathbf{sSet}_{K/}$ or $\mathbf{sSet}_{L/}$. The functors $K \star -$ and $- \star L$ preserve colimits.

Example 2.4.

1. For the standard simplices we have $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}$, for $i, j \geq 0$. The isomorphism is compatible with the inclusions.
2. Let $K \in \mathbf{sSet}$, the **right cone** or **cocone** $K^{\triangleright} = K \star \Delta^0$, and the **left cone** or **cone** $K^{\triangleleft} = \Delta^0 \star K$.

With the over and undercategories at hand, the colimit and limit is easy to define (It might be intuitive to think the simplicial set K as a diagram category):

Definition 2.5 (1.2.13.4 in[HTT])**.** Let $\mathcal{C}$ be an ∞-category and let $p : K \to \mathcal{C}$ be an arbitrary map of simplicial sets. A **colimit** for p is an initial object of $\mathcal{C}_{p/}$, and a **limit** for p is a final object of $\mathcal{C}_{/p}$.

3 Fibrations and Grothendieck construction

Sometimes we would like to build a category out of the data of a functor, the most straight forward way of doing this is "summing" up the fiber of each target. As in the special case let $\chi : \mathcal{D} \to \mathbf{Gpd}$ be a functor, then for each $D \in \mathcal{D}$, there is a groupoid $\chi(D)$ associate to it.

Definition 3.1 (2.1.1 beginning in [HTT])**.** For a functor $\chi : \mathcal{D} \to \mathbf{Gpd}$, the **Grothendieck construction** gives rise to a new category $\mathcal{C}_\chi$ where

- objects: pairs (D, a), with $D \in \mathcal{D}$ and $a \in \chi(D)$
- morphisms: pairs (f, α) where f is a morphism between D and D', and $\alpha : \chi(f)(a) \to a'$ is an isomorphism in the groupoid $\chi(D')$

Suppose we are given a functor between ∞-categories $\mathcal{C} \xrightarrow{p} \mathcal{D}$, we can construct another functor eating the points in $\mathcal{D}$ and spit out the fiber $\mathcal{C}_D := p^{-1}(D)$. It is not guaranteed that we will land in **Gpd**, actually not even $\mathbf{Cat}_\infty$. We need to put some restrictions on p to make the landing space good enough.

Definition 3.2. A **Kan fibration** is a patten $K \xrightarrow{p} L$ in **sSet** such that for $0 \leq i \leq n$, we have the lifting

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & L \end{tikzcd}$$

$K \to L$ is a **Kan equivalence** if $|K| \to |L|$ is a weak equivalence in **Top** after geometric realization.

Remark 3.3 (also a definition)**.** Usually a Kan fibration is the best thing you can expect, because the **trivial fibration** (where you get the lifting with respect to the inclusion of the boundry $\partial\Delta^n$ instead of horns) is way harder to get. Making the restriction even looser we get various of fibrations:

- If the condition in previous definition changes to $0 \leq i < n$, we call $K \xrightarrow{p} L$ **left fibration**.
- If it changes to $0 < i \leq n$, we call $K \xrightarrow{p} L$ **right fibration**.
- If it changes to $0 < i < n$, we call $K \xrightarrow{p} L$ **inner fibration**.

The names basically refer to which outter horns are included in the lifting property. We consider the functor arise from them and see the target can be described in a relative nice way. (There won't be a proof here though)

Proposition 3.4. *It is an easy observation that Kan fibration implies left fibration implies inner fibration.*

- *If $\mathcal{C} \xrightarrow{p} \mathcal{D}$ is an inner fibration, then the fiber of $D \in \mathcal{D}$ $\mathcal{C}_D$ is an ∞-category. In another word, the assignment χ defined by sending D to $\mathcal{C}_D$ is a functor from $\mathcal{D}$ to* **Cat**$_\infty$*.*
- *If $\mathcal{C} \xrightarrow{p} \mathcal{D}$ is a left(right) fibration, then the assinment χ defined by sending D to $\mathcal{C}_D$ is a functor from $\mathcal{D}$ to* **Gpd***.*

The next step is, can we come back a fibration via the Grothendieck construction, if given a functor χ from $\mathcal{D}$ to **Gpd** or **Cat**$_\infty$. It turns out that the answer is yes for the former but not the latter.

Why? Here are some observations, if $\mathcal{C} \xrightarrow{p} \mathcal{D}$ is a left fibration, then every edge $D \xrightarrow{f} D'$ gives a functor on their fibers $\mathcal{C}_D \xrightarrow{F} \mathcal{C}_{D'}$. This is because all the arrows are isomorphisms in a groupoid, so you don't have the chance to go wrong with the composition to make F a functor.

But this is not true for inner fibration $\mathcal{C} \xrightarrow{p} \mathcal{D}$. An edge $D \xrightarrow{f} D'$ will give rise to a correspondence between their fibers, which is a little weaker than a functor. However the Grothendiece construction will always give you a functor between fibers, since the information encoded in the definition exactly guarantee the composition to work out. So we get something actually better than an inner fibration from applying the Grothendieck construction to a functor $\chi : \mathcal{D} \to$ **Cat**$_\infty$.

We get something called a **coCartesian fibration**, the definition is tedious and essentially saying the same thing: this is the weakest condition to guarantee that an edge will lift to a functor between their fibers.

Definition 3.5 (2.4.1.1 in [HTT]). Let $p : K \to L$ be an inner fibration of simplicial sets. Let $f : x \to y$ be an edge in K. We say that f is **p-Cartesian** if the induced map

$$K_{/f} \to K_{/y} \times_{L_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration. We will say that f is **p-coCartesian** if it is Cartesian with respect to the morphism $p^{op} : K^{op} \to L^{op}$.

Definition 3.6 (2.4.2.1 in [HTT]). We will say that a map $p : K \to L$ of simplicial sets is a **Cartesian fibration** if the following conditions are satisfied:

- the map p is an inner fibration.
- for every edge $f : x \to y$ in L and every vertex $\tilde{y}$ of K with $p(\tilde{y} = y$, there exists a p-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ with $p(\tilde{f}) = f$.

We say that p is a **coCartesian fibration** if the opposite map $p^{op} : K^{op} \to L^{op}$ is a Cartesian fibration.

We will not elaborate the definition, instead we are going to see an example that showed up in Foling's talk earlier, to see why this is important.

Example 3.7 (2.0.0.7 in [HA] as an example). A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : C^{\otimes} \to$ **nFin** with the following property:

(*) For each $n \geq 0$, the innert morphisms $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \to (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

Why the assignment is required to be coCartesian is, for each edge (innert morphisms) we would like them to induce functors between layers.

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