HTPY PULLBACK IN GPD 2

LI, ANG

1. GETTING A GROUPOID FROM A DISCRETE GROUP

Definition 1.1. G is a discrete group, define a groupoid $*//G$ by:

- obj: $*$
- mor: $\mathrm{Hom}(*,*) \cong G$

This gives an assignment from **Gp** to **Gpd**.

Remark 1.2. We carefully use the terminology "Eilenberg-Mac Lane category", denote $*//...//G$ $(n-times)$ to be the category that satisfies

- obj: $*$
- k-morphisms: $\mathrm{Hom}(*_{k-1}, *_{k-1}) \cong *$ for $k \neq n$
- $\mathrm{Hom}(*_{n-1}, *_{n-1}) \cong G$

Question 1.3. *If we consider groups as* $(1,0)$*-categories,* $G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} H$ *gives* $*//G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} *//H$*. But when are* φ *and* ψ *naturally iso.?*

Answer. The natural iso.s are given in squares

$$\begin{array}{ccc} *_H & \xrightarrow{h} & *_H \\ {\scriptstyle \psi(g)}\downarrow & & \downarrow{\scriptstyle \varphi(g)} \\ *_H & \xrightarrow{h} & *_H \end{array}$$

So φ and ψ are naturally iso. if $\exists\, h$ s.t. $\psi(g) \cdot h = h \cdot \varphi(g)$. In other words for any g, $\varphi(g) = h \cdot \psi(g) \cdot h$.

Example 1.4. Describe $|\,\mathrm{Fun}(*//\mathbb{Z}, *//G)|$.

$$\mathrm{Fun}(*//\mathbb{Z}, *//G) \cong \begin{cases} obj : homo.from\ \mathbb{Z}\ to\ G \cong G \\ mor : conjugations \end{cases}$$

So, $\pi_0(\mathrm{Fun}(*//\mathbb{Z}, *//G)) \cong$\{conj. classes\}, $\pi_1(\mathrm{Fun}(*//\mathbb{Z}, *//G))$ at $g \cong \mathrm{Aut}_{\mathrm{Fun}(*//\mathbb{Z}, *//G)}(g) \cong C(g)$ (centralizer of $g \in G$).
So we get free loop space: $\mathcal{LBG} \cong |\,\mathrm{Fun}(*//\mathbb{Z}, *//G)| \cong \coprod_{[g]} | * //C(g)|$.

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Example 1.5. Now describe $\mathcal{L}^2\mathcal{BG}$=$\{S^1 \times S^1 \xrightarrow{\text{LI, ANG}} BG\}$.
$\mathcal{L}^2\mathcal{BG} \cong |\operatorname{Fun}(*/\!/\mathbb{Z}, |\operatorname{Fun}(*/\!/\mathbb{Z}, */\!/G)|)| = |\operatorname{Fun}(*/\!/\mathbb{Z}, \coprod_{[g]} */\!/C(g))|$
$= \coprod_{[g]} \coprod_{[h \in C(g)]} |*/\!/C_{C(g)}(h)|$, where $C_{C(g)}(h) = \{k \in G | hk = kh, gk = kg\}$.
Or by adjunction, $\operatorname{Fun}(*/\!/\mathbb{Z}, \operatorname{Fun}(*/\!/\mathbb{Z}, */\!/G)) \cong \operatorname{Fun}(*/\!/\mathbb{Z} \times */\!/\mathbb{Z}, */\!/G)$, so $\{S^1 \times S^1 \to BG\} = \pi_0(\operatorname{Hom}(*/\!/(\mathbb{Z} \times \mathbb{Z}), */\!/G)) = \operatorname{Hom}(\mathbb{Z}^2, G)/\sim$, where $(g, h) \sim (kgk^{-1}, khk^{-1})$

Remark 1.6. If your space has htpy 1-type only(only π_0 and π_1), you can calculate htpy pullback in **Gpd**.

2. FORMULA FOR HTPY PULLBACK IN GROUPOID

Notation 2.1. Usually we write BG for $*/\!/G$.

Formula 2.2. If G, H, K are groupoids, and we have

$$\begin{array}{ccc} H \times_G K & \longrightarrow & K \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ H & \xrightarrow{\psi} & G \end{array}$$

Then $H \times_G K$ is given by:

- obj: (k, h, τ), where k, h are objects in K, H, and τ is an arrow in G connecting $\varphi(k)$ and $\psi(h)$.
- mor: $(k \xrightarrow{\alpha} k', h \xrightarrow{\beta} h', \tau \Rightarrow \tau' | \tau'\varphi(\alpha) = \psi(\beta)\tau)$, i.e.

$$\begin{array}{ccc} \varphi(k) & \xrightarrow{\varphi(\alpha)} & \varphi(k') \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau'} \\ \psi(h) & \xrightarrow{\psi(\beta)} & \psi(h') \end{array}$$

Example 2.3. In the case $G = BG$, $H = BH$ and $K = BK$, we will get double coset.
Consider

$$\begin{array}{ccc} BH \times_{BG} BK & \longrightarrow & BK \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ BH & \xrightarrow{\psi} & BG \end{array}$$

The htpy pullback $BH \times_{BG} BK$ is given by:

- obj: $\{(*, *, g)\} \cong G$.
- mor: $\{(k, h) \mid h \cdot g_1 = g_2 \cdot k\}$

So as long as $g_1 \in Hg_2K$, there are arrows connecting them. $\pi_1(BH \times_{BG} BK) = \operatorname{Hom}((*, *, g), (*, *, g))$ should have the same size of the double coset g is in, which is $[H : H \cap gKg^{-1}]|K|$ or $|H|[K : K \cap g^{-1}Hg]$

Remark 2.4. More summarized, we have an adjoint pair between $(1, 0)$-cats and $(\infty, 0)$-cats. (An (n, r)-cat $\mathscr{C}$ indicating all m-morphisms in $\mathscr{C}$ are trivial for $m > n$, and all k-morphisms in $\mathscr{C}$ are invertible for $k > r$)

$$\{(\infty, 0) - cats\} \underset{i}{\overset{\tau_{\leq 1}}{\rightleftarrows}} \{(1, 0) - cats\}$$

where $\tau_{\leq 1}$ is the truncation and i is the imbedding. i is a right adjoint (also fully faithful) thus preserves limit (and htpy pullback is a limit). So compare the formula for spaces and for groupoids you would feel some similarities.

3. A rough idea about constructing 2-Groupoid

First notice that 2-morphisms would still forms an abelian group by Eckmann-Hilton argument. So if given a group G and an abelian group A as π_1 and π_2, what kind of 2-groupoid $\mathscr{D}$ can we construct? Or in other words, what kind of extra data is required?

Let's start with the easiest case. Assume G acts on A trivially both on the left and the right. From our discussion on 1-groupoid, it is reasonable to set $\mathrm{Hom}(*,*) \cong G$ and $\mathrm{Hom}(id_G, id_G) \cong A$. In general, a 2-morphism set $\mathrm{Hom}(g_1, g_2)$ for some $g_1, g_2 \in G$ might be gained from applying left and right action of G on $\mathrm{Hom}(id_G, id_G)$, which is isomorphic to A. Since G acts trivially on A, we have for every $g_1, g_2 \in G$, $\mathrm{Hom}(g_1, g_2) \cong A$. In this case, our 2-groupoid $\mathscr{D}$ with $\pi_1 \cong G$ and $\pi_2 \cong A$ is just the product category of two Eilenberg-Mac Lane categories $*//G$ and $*//*//A$.

However, if G acts nontrivially on A, we don't have a reason to expect $\mathrm{Hom}(g_1, g_2)$ to be isomorphic to A for every $g_1, g_2 \in G$, it might not even be a group. We might want to look at cross modules, which has easy homotopy groups and encodes algebraic data.

References

[Noohi] B. Noohi. *Notes on 2-Groupoids, 2-Groups and Crossed Modules.* Available at `http://www.maths.qmul.ac.uk/~noohi/papers/Notes(corrected).pdf`.