Equivariant spheres, Freudenthal Suspension theorem and the category of naive spectra

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1 Motivation

Equivariantly we care about G-spaces and G-spectra, the integer grading is not sufficient to encode the information of the group actions thus people enlarge the index set in order to fix this deficiency. That is why we would like to have this $RO(G)$-graded system on spheres.

Furthermore one need the spheres to be equipped with a G action to talk about the equivariant Poincare duality. If otherwise, let $E^*(-)$ and $E_*(-)$ be a general cohomology and homology theory (coefficient systems actually) respectively, we would like to expect for a G-manifold X has dimension n and embeds into some $\mathbb{R}^m$ with a G-action, we can identify

$$E^n(X) \cong E_{m-n}(X)$$

However, roughly $E^n(X) = [X, E]^G_{-n}$ and $E_{m-n}(X) = [S, \mathrm{colim}(E_{m-n} \wedge X)]^G = [S, \mathrm{colim}(E_{m-n} \wedge X)^G]$. Notice that X is at different sides of the homotopy classes of G-mapping spaces, and we can not get along with this provided only integer grading.

So let us start our journey with another family of spheres: indexed over representations. **In order to be safe we will assume G to be a finite group throughout the note even sometimes might not necessary.**

2 Representation and Universe

2.1 Representation of a Group

Definition 2.1. A representation of a group G on a vector space V over a field k is a group homomorphism

$$\varphi : G \to GL(V)$$

where $GL(V)$ is the general linear group on V. Here V is called the representation space and the dimension of V is the dimension of the representation.

We usually refer to V as the representation itself if there is no confusion. The reason why we are introducing this algebraic gadget is, a group homomorphism $\varphi : G \to GL(V)$ is the same data as a group action $\psi : G \times V \to V$.

In practice it is convenient to restrict the codomain $GL(V)$ to the orthogonal matrices $O(V)$ by the process of Gram-Schmidt. We thus call the representations orthogonal representations. And most of the case our field k will be the real number $\mathbb{R}$.

Example 2.2.

1. trivial representation: φ is mapping everything in G to the identity matrix in $GL(V)$.
2. (left) regular representation: For a finite group G, the (left) regular representation ρ is a representation on the k-vector space V generated by the elements of G. i.e. $V = \mathbb{R}[G] = \bigoplus_{g \in G} \mathbb{R}$, the group ring.

Apparently these examples are not down to earth enough, so let's also look at some examples of examples:

Example 2.3.

1. for $G = C_2 = \{e, \sigma\}$, the cyclic group of order 2, the regular representation V over $\mathbb{R}$ is a real vector space with a basis $\{e, \sigma\}$. The group homomorphism φ is given by

$$\varphi(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \varphi(\sigma) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2. for $G = C_3 = \{e, \sigma, \sigma^2\}$, the cyclic group of order 3, the regular representation V over $\mathbb{R}$ is a real vector space with a basis $\{e, \sigma, \sigma^2\}$. The group homomorphism φ is given by

$$\varphi(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \varphi(\sigma) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ \varphi(\sigma^2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise 2.4. Show that for a prime p, the cyclic group C_p of order p has regular representation V generated by the p-th roots of unity.

For a vector space, we can talk about its subspace. Same thing with G-actions.

Definition 2.5. Let V be a G-representation given by $\varphi : G \to GL(V)$. A linear subspace $W \subseteq V$ is called a G-invariant if $\varphi(g)w \in W$ for all $g \in G$ and all $w \in W$. The restriction of φ to a G-invariant subspace W is known as a subrepresentation.

All representations have a subrepresentation with the trivial G-invariant subspaces, but sometimes we can break down a whole representation into non-trivial pieces until we can't do it anymore. Those can not further break down give the idea of irreducible representation.

Definition 2.6. A representation $\varphi : G \to GL(V)$ is said to be irreducible if it has only zero subrepresentations.

We have learned the fact that we can decompose a representation into irreducible subrepresentations. On the other hand, does every representation V sitting inside a "larger" representation? The answer is Yes.

Proposition 2.7. *Let G be a finite group and ρ be its regular representation. For any representation V of G, there exists an integer n such that V embeds in ρ^n.*

Proof. The proof is pretty algebraic, we sketch it in 2 steps:

1. Any representation V can be decomposed into irreducible representations, i.e. $V \cong \bigoplus_i n_i V_i$ where V_i is irreducible.
2. Let V be an irreducible representation, then V embeds in ρ.

The first step is straightforward. As to the second one, V can be thought of as a $\mathbb{R}[G]$-module, where $\mathbb{R}[G]$ is the group ring. If V is irreducible then it is a simple $\mathbb{R}[G]$-module (the only proper submodule is 0). The Maschke's theorem gives us the quotient from the free $\mathbb{R}[G]$-module $\mathbb{R}[G]^n$ to V has a splitting:

$$\mathbb{R}[G]^n \leftarrow V$$

Since V is simple, it should embed into one copy of $\mathbb{R}[G]$, otherwise it would split further into direct sum of non-trivial submodules, contradicts its simplicity. ∎

Actually, over the complex number $\mathbb{C}$ for any dimension d irreducible G-representation V, we have

$$\rho \cong V^d \oplus W$$

where ρ is the regular representation and W doesn't contain V as a subrepresentation. This is not true over $\mathbb{R}$.

2.2 Universes of G-representation

For the integer grading, we write $\mathbb{Z}$ as the set of all integers. Now we need a parallel concept, i.e. a collection of all the representations we care about.

Definition 2.8. A G-universe U is a countable direct sum of representations such that U contains

- the trivial representation;
- each of its subrepresentations infinitely often.

Usually we would expect the universe contains all the "small blocks", i.e. the irreducible representations.

Definition 2.9. A G-universe U is complete if it contains every irreducible representations up to isomorphism.

A complete universe is good because it guarantees the existance of all transfers (they are wrong way maps make sense only stably, we will see more in later talks):

$$G/H \to G/K, \text{ for } K \subseteq H$$

Example 2.10. Due to the fact that every representation embeds into some copies of the regular representation, a common universe we are taking is $U = \rho^\infty$. And it is complete.

Exercise 2.11. Give an example of an incomplete universe of C_2, is it trivial?

3 G-spheres and the Equivariant Freudenthal suspension theorem

We talked about the representations of a group, of course they have a G-action. But where are the spheres? We are going to take good use of the one point compactification.

3.1 G-spheres

Let V be a G-representation, let S^V denote the one point compactification of V. Observe that G acts trivially on the point ∞, which is consider as the based point of the G-space S^V. Alternatively, S^V can be thought as $D(V)/S(V)$, where $D(V)$ is the unit disk and $S(V)$ the unit sphere of V.

S^V inherits the G action from the representation V. With this family of G-spheres at hand, we can define the equivariant suspention and the equivaraint loop funtors:

$$\Sigma^V X = S^V \wedge X \quad \Omega^V X = Map_*(S^V, X)$$

(Here G acts on $Map_*(S^V, X)$ by conjugation.) And they form an adjunction:

$$\Sigma^V : G\mathbf{Top} \underset{\perp}{\rightleftarrows} G\mathbf{Top} : \Omega^V$$

Exercise 3.1. Prove this.

3.2 connectivity and the suspension theorem

We relate the connectivity of a G-space X to the connectivity of its fixed points.

Definition 3.2. Let X be a G-space. For H a subgroup of G, define an natural number-valued function

$$c^H(X) = \text{the connectivity of } X^H$$

Remark 3.3. The convention is when X is empty, the function value will be extended to -1. The connectivity of non-path connected spaces is also -1.

We can talk about the G-map being a ν-equivalence, where ν is an natural number-valued function with all subgroup H of G as input.

Definition 3.4 (I.3 in [1] right above Theorem 3.1)**.** Let $f : X \to Y$ be a G-map between G-spaces. We say that f is a ν-equivalence if

$$f^H : X^H \to Y^H$$

is a $\nu(H)$-equivalence for every H a subgroup of G.

Theorem 3.5 (IX.1 Theorem 1.4 in [1]). *[equivariant Freudenthal suspension theorem] The map $\eta_Y : Y \to \Omega^V \Sigma^V Y$ is a ν-equivalence if ν satisfies:*

- $\nu(H) \leq 2c^H(Y) + 1$ *for all subgroup H with $V^H \neq 0$;*
- $\nu(H) \leq c^K(Y)$ *for all $K \subseteq H$ with $V^K \neq V^H$.*

Therefore the suspension map

$$\Sigma^V : [X, Y]_G \to [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if $dim(X^H) \leq \nu(H)$ for all H, and bijection if $dim(X^H) \leq \nu(H) - 1$.

Note that the non-equivariant Freudenthal suspension theorem can be viewed as a special case, by letting $\nu : point \to \mathbb{N}$ to be the constant function, i.e. a chosen natural number. Let's look at a simple example.

Example 3.6. Let $G = C_2$, $V = \rho$ the regular representation, and the C_2-space be S^2 with antipodal action, therefore $(S^2)^{C_2} = \varnothing$ and $(S^2)^e = S^2$. Then ν needs to satisfy the following condition in order for $\eta : S^2 \to \Omega^\rho \Sigma^\rho S^2$ to be a ν-equivalence:

- $\nu(C_2) \leq 2c^{C_2}(S^2) + 1 = -1$, $\nu(e) \leq 2c^e(S^2) + 1 = 3$, since $\rho^e = \mathbb{R}^2 \neq 0$ and $\rho^{C_2} = \mathbb{R} \neq 0$;
- $\nu(C_2) \leq c^e(S^2) = 1$, since $\rho^{C_2} = \mathbb{R} \neq \mathbb{R}^2 = \rho^e$.

This is telling us the map $\eta^e : S^2 \to \Omega^\rho \Sigma^\rho S^2$ is a 3-equivalence, it gives information about $\Omega^\rho \Sigma^\rho S^2$, which is very hard to visualize.

There is another version of Freudenthal using coefficient systems, using Mackey functors (see XI.4 Theorem 4.5 in [1]), I am not sure which is the better version to present.

4 $RO(G)$-graded homotopy groups and stable homotopy groups of G-spaces

4.1 Unstable homotopy group of G-spaces

Now we have the G-spheres, passing to homotopy classes we can talk about homotopy groups graded by representations.

Definition 4.1. Let V be a G-representation. For any H subgroup of G, one may think V as an H-representation as well. Define

$$\pi_V^H(X) = [S^V, X]_H \cong [G_+ \wedge_H S^V, X]_G$$

These are called the $RO(G)$-graded homotopy groups of a G-space X.

Remark 4.2. Inside $RO(G)$-grading, there is an issue of choosing an isomorphism as a representative between $S^V \wedge S^W$ and $S^{V \oplus W}$. It is possible to make such a choice such that the $RO(G)$-graded homotopy group does not depent on the chosen isomorphism. This discussion can be found in Lewis and Mandall's paper *Equivariant universal coefficient and Künneth spectral sequences* appendix A.

Remark 4.3. There is a warning to make here that $\pi_V^H(X)$ is not actually a group, the pinch map $S^V \to S^V \vee S^V$ is equivariant, since the equater is G-invariant (because the action preserves distance). But the base point doesn't behave well under this collapsing. Moreover to get the inverse, we need one copy of the trivial representation anyway to obtain the negative orientation.

Example 4.4.

1. Let V be the trivial representation of dimension n, then $\pi_n^H(X) = [S^n, X]_H \cong [S^n, X^H] = \pi_n(X^H)$, we get the integer graded equivariant homotopy group (coefficient system) back;
2. Let $G = C_2$, $V = \sigma$ the sign representation. $\pi_\sigma^e(X) = [S^\sigma, X]_e = \pi_1(X)$ and $\pi_\sigma^{C_2}(X) = [S^\sigma, X]_{C_2}$;
3. Let $G = C_2$, $V = \rho$, then $\pi_\rho^e(X) = [S^\rho, X]_e = \pi_2(X)$ and $\pi_\rho^{C_2}(X) = [S^\rho, X]_{C_2}$;
4. From above one might guess that for a G-representation V of dimension d, $\pi_V^e(X) = [S^V, X]_e = \pi_d(X)$.

4.2 Stable homotopy group of G-spaces

Like in non-equivariant case, we can define stable maps between G-spaces X and Y by suspending enough times. However now the grading is over G-representations, we should be more careful when taking colimit.

Definition 4.5. Let U be a complete G-universe for a finite based G-CW complex X and any based G-space Y. Define the stable G-map between X and Y

$$\{X, Y\}_G = \operatorname*{colim}_{V \in U} [\Sigma^V X, \Sigma^V Y]_G$$

where the colimit is taken over

$$[\Sigma^V X, \Sigma^V Y]_G \xrightarrow{-\wedge S^{W-V}} [\Sigma^W X, \Sigma^W Y]_G$$

for any $V \subseteq W$.

Since X is compact, recall we have adjunction $\Sigma^V \dashv \Omega^V$, thus

$$\operatorname*{colim}_{V \in U} [\Sigma^V X, \Sigma^V Y]_G \cong \operatorname*{colim}_{V \in U} [X, \Omega^V \Sigma^V Y]_G$$

Usually this is preferred because it saves us to think about two spots at the same time.

Remark 4.6. When X is not finite CW, we need extra work to define such stable G-maps.

In the case G is finite, we do have this stabilized at some representation.

Proposition 4.7. *If G is finite and X is finite dimensional, there exists a representation V_0 such that for any representation V,*

$$[\Sigma^{V_0}X, \Sigma^{V_0}Y]_G \xrightarrow{\cong} [\Sigma^{V_0 \oplus V}X, \Sigma^{V_0 \oplus V}Y]_G$$

is an isomorphism.

In other words,

$$\{X, Y\}_G \cong [\Sigma^{V_0}X, \Sigma^{V_0}Y]_G$$

Let X be the G-sphere S^V, we get the $RO(G)$-graded stable homotopy group of a G-space

$$\pi_V^{Stab,G}(Y) = \{S^V, Y\}_G = \operatorname*{colim}_{V \in U}[\Sigma^V S^V, \Sigma^V Y]_G \cong \operatorname*{colim}_{V \in U}[S^V, \Omega^V \Sigma^V Y]_G$$

4.3 Equivariant stabilization and the naive G-spectra

Non-equivariantly we have the stablization functor $Q = \operatorname*{colim}_n \Omega^n \Sigma^n = \Omega^\infty \Sigma^\infty$, here we define the equivariant stablization functor by changing the way of taking colimit.

Definition 4.8. Let U be a complete G-universe, for a G-space X, define

$$QX := \operatorname*{colim}_{V \in U} \Omega^V \Sigma^V X$$

where the colimit is taken over

$$\Omega^V(\Sigma^V X) \xrightarrow{\Omega^{W-V} \circ (- \wedge S^{W-V})} \Omega^W \Sigma^W X$$

for any $V \subseteq W$.

Remark 4.9. Since Σ^V and Ω^V do not commute, the order of the composition above matters.

The functor Q is defined to be a colimit, thus has the universal properties.

Proposition 4.10. *For any G-representation $V \in U$, let X be a G-space, then there is a natural homeomorphism*

$$QX \cong \Omega^V Q \Sigma^V X$$

Exercise 4.11. Prove this.

Now let's think naively, non-equivariantly to get a spectrum we need the structure maps $S^1 \wedge X_n \to X_{n+1}$ for all n. If we do the same thing, we will get a version of G-spectra where the structure maps don't really interfere with the G-action.

Definition 4.12. A naive G-spectrum contains the following data:

- A family of G-spaces X_n for $n \in \mathbb{N}$;
- A family of G-maps from $S^1 \wedge X_n \to X_{n+1}$ for all n as the structure maps.

Later we will see a right notion of G-spectra (the genuine G-spectra) where the structure maps are from $S^V \wedge X_W$ to $X_{V\oplus W}$, i.e. $RO(G)$-graded, with V and W in some given universe U. Providing this, we can also think the naive G-spectra as the genuine G-spectra where the universe U contains only the trivial representation.

We still have this "Σ^∞" functor

$$\Sigma^\infty_G : G\mathbf{Top} \to G\mathbf{Sp}$$

given by $(\Sigma^\infty_G X)_n = S^n \wedge X$. Choose an universe U and let $V \in U$, we can actually let $(\Sigma^\infty_G X)_V = S^V \wedge X$, thus the target would land in the category of genuine G-spectra.

Now let's step back and take a look at an example of the equivariant stable maps.

Example 4.13. For a finite based G-CW complex X and any based G-space Y, if U is the trivial universe, we have $\{X, Y\}_G = \operatorname{colim}_n [\Sigma^n X, \Sigma^n Y]_G$, the action of taking colimit has nothing to do with the G-action. So this is the same as homotopy classes of maps between the naive G-spectra $\Sigma^\infty_G X$ and $\Sigma^\infty_G Y$.

Exercise 4.14. I'll buy you a beer if you do another example.

5 A general approach of the Universal property of stabilization

We saw the construction and the universal property of the category of spectra non-equivariantly in 1.3 and 1.4, and the construction works equivariantly too, without doing much modifying, especially on the way of getting the catefory of naive spectra. Here in this section we are going to see another way to approach it, via localizing at a single morphism in a symmetric monoidal category.

Let $\mathcal{C}$ be a symmetric monoidal category, we can define another category using the data of $\mathcal{C}$.

Definition 5.1 (2.0.0.1 in [5]). Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We define a new category $\mathcal{C}^\otimes$ as follows:

- object: finite sequence of objectis of $\mathcal{C}$, denote by $[C_1, ..., C_n]$.
- morphism from $[C_1, ..., C_n]$ to $[C'_1, ..., C'_m]$ in $\mathcal{C}^\otimes$ consists of a subset $S \subseteq \{1, ..., n\}$, a map of finite sets $\alpha : S \to \{1, ..., m\}$, and a collection of morphisms $\{f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1\le j\le m}$ in the category $\mathcal{C}$.
- composition: suppose we are given $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ and $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_k]$ in $\mathcal{C}^\otimes$, determining subsets $S \subseteq \{1, ..., n\}$ and $T \subseteq \{1, ..., m\}$ with maps $\alpha : S \to \{1, ..., m\}$ and $\beta : T \to \{1, ..., k\}$. The composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, ..., n\}$, the composition map $\beta \circ \alpha : U \to \{1, ..., k\}$, and for $1 \le l \le k$ we have

$$\bigotimes_{(\beta\circ\alpha)(i)=l} C_i \simeq \bigotimes_{\beta(j)=l} \bigotimes_{\alpha(i)=j} C_i \to \bigotimes_{\beta(j)=l} C'_j \to C''_l$$

Inside $\mathcal{C}$ we can consider an invertible object: X such that there exists $X^* \in \mathcal{C}$ and $X \otimes X^* \cong id_{\mathcal{C}} \cong X^* \otimes X$. Let $\mathbf{Cat}_\infty$ be the category of infinite categories, then $\mathcal{C}^\otimes$ is an object in $\mathbf{CAlg}(\mathbf{Cat}_\infty)$. Let $\mathbf{Mod}_{\mathcal{C}^\otimes}(\mathbf{Cat}_\infty)$ be the category where the objects are ∞-categories $\mathcal{M}$ that are $\mathcal{C}^\otimes$-modules (i.e. have a $\mathcal{C}$-action $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$). It is a fact that $\mathbf{CAlg}(\mathbf{Mod}_{\mathcal{C}^\otimes}(\mathbf{Cat}_\infty)) \cong \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/}$.

On one side we consider the full subcategory $\mathbf{CAlg}(\mathbf{Cat}_\infty)^X_{\mathcal{C}^\otimes/} \subseteq \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/}$, which objects are monoidal maps $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ begining with $\mathcal{C}^\otimes$ and sending X to an invertible object in $\mathcal{D}$. On the other side, let $\mathcal{S}_X$ to be the collection of morphisms in $\mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/}$:

$$\begin{tikzcd} & \mathcal{C}^\otimes \arrow[dl] \arrow[dr] & \\ Free_{\mathcal{C}^\otimes}(\mathcal{C}) \arrow[rr, "Free_{\mathcal{C}^\otimes}(X\otimes -)"'] & & Free_{\mathcal{C}^\otimes}(\mathcal{C}) \end{tikzcd}$$

This is a single morphism collection, and $Free^{\otimes}_{\mathcal{C}}(-)$ is the left adjoint functor of the forgetful functor $U : \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/} \cong \mathbf{CAlg}(\mathbf{Mod}_{\mathcal{C}^\otimes}(\mathbf{Cat}_\infty)) \to \mathbf{Mod}_{\mathcal{C}^\otimes}(\mathbf{Cat}_\infty)$. If we consider the category $\mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/}$ localized at $\mathcal{S}_X$, we see that it is actually the same as $\mathbf{CAlg}(\mathbf{Cat}_\infty)^X_{\mathcal{C}^\otimes/}$.

Proposition 5.2 (part of Theorem 2.1 in [Rob]). *There is a categorical equivalence*

$$\mathbf{CAlg}(\mathbf{Cat}_\infty)^X_{\mathcal{C}^\otimes/} \cong \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes/}[\mathcal{S}_X^{-1}]$$

Let $\mathbf{Pr}^L$ be the full subcategory of all the representable ∞-categories, then the inclution $\mathbf{CAlg}(\mathbf{Cat}_\infty)^X_{\mathcal{C}^\otimes} \hookrightarrow \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathcal{C}^\otimes}$ has a left adjoint, and the following square commutes:

$$\begin{tikzcd} \mathbf{CAlg}(\mathbf{Pr}^L)^X_{\mathcal{C}^\otimes/} \arrow[d, "U"'] & \mathbf{CAlg}(\mathbf{Pr}^L)_{\mathcal{C}^\otimes/} \arrow[l, "L"'] \arrow[d, "U"] \\ \mathbf{Mod}_{\mathcal{C}^\otimes[X^{-1}]}(\mathbf{Pr}^L) & \mathbf{Mod}_{\mathcal{C}^\otimes}(\mathbf{Pr}^L) \arrow[l, "\mathcal{C}^\otimes[X^{-1}]\otimes -"] \end{tikzcd}$$

The bottom line is the change of ring functor, in particular sending $\mathcal{C}$ to $\mathcal{C}[X^{-1}]$. This construction is universal due to the previous proposition and the universal property of localization.

Remark 5.3. However inverting an object in $\mathcal{C}^\otimes$ is not as clean and neat as you think. We will not talk about the definition of the category $\mathcal{C}^\otimes[X^{-1}]$ since the change of ring functor works analogously. See Definition 2.6 (2.36) in [Rob] for details.

How do we rain down these abstract clouds? In the case we care, let $\mathcal{C}$ to be **Top** or G**Top**$_*$ with smash product $\wedge$ and unit S^0, let X to be S^1 with trivial G-action, then $\mathcal{C}[X^{-1}]$ will be **Sp** or G**Sp**, respectively. But hold on, this is not at all easy to see. We need further assume X to be symmetric object.

Definition 5.4 (Definition 2.16 (2.93) in [Rob]). Let $\mathcal{C}$ be a symmetric monoidal category and let X be an object in $\mathcal{C}$. We say that X is symmetric if there is a 2-equivalence in $\mathcal{C}$ between the cyclic permutation (123) and the identity. i.e. we would like the existence of a 2-cell as a homotopy between (123) and the identity.

$$\begin{tikzcd} X \otimes X \otimes X \arrow[r, "(123)"] \arrow[d, "id"'] & X \otimes X \otimes X \\ X \otimes X \otimes X \arrow[ur, "id"'] & \end{tikzcd}$$

Here is a hint why the 3-cycle is considered. Let's try to plug in X to be S^1 in **Top**, then we know the switch map from $S^1 \wedge S^1$ to itself has degree -1, thus can not be homotopic to the identity. This issue is fixed by considering the 3-cycle (123), as it decomposes into two 2-cycles.

If we have X to be symmetric (which we do, since (123) on $S^1 \wedge S^1 \wedge S^1$ is already homotopic to the identity in spaces), then we can find the connection between $\mathcal{C}[X^{-1}]$ and $Stab_X(\mathcal{C})$, where for a $\mathcal{C}^{\otimes}$-module $\mathcal{M}$, $Stab_X(\mathcal{M})$ is defined as the colimit of the sequence

$$... \xrightarrow{X\otimes -} \mathcal{M} \xrightarrow{X\otimes -} \mathcal{M} \xrightarrow{X\otimes -} \mathcal{M} \xrightarrow{X\otimes -} ...$$

Theorem 5.5 (Corollary 2.22 (2.106) in [Rob]). *Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal ∞-category and let X be a symmetric object in $\mathcal{C}$. Given a $\mathcal{C}^{\otimes}$-module $\mathcal{M}$, we have*

$$\mathcal{C}^{\otimes}[X^{-1}] \otimes \mathcal{M} \to Stab_X(\mathcal{M})$$

is an equivalence. In particular, $\mathcal{C}^{\otimes}[X^{-1}]$ is equivalent to the stablization $Stab_X(\mathcal{C}^{\otimes})$.

We need one more brick to build the bridge from the stabilization mentioned in previous talks, in which $\mathbf{Sp}(\mathcal{C})$ for a presentable ∞-category $\mathcal{C}$ is defined as $\mathbf{Exc}_*(\mathcal{S}_*^{\mathbf{fin}}, \mathcal{C})$ in [5], 1.4.2.8. (And let us move things from 1-**Cat** to $\mathbf{Cat}_\infty$, that is to say $\mathbf{Top}_*$ is $\mathcal{S}_*$ now, plus we need a version of the ∞-category of G-spaces. It turns out that $S_*^G := \mathrm{Fun}(\mathcal{O}rb_G^{op}, \mathcal{S}_*)$ will satisfy the properties we want.)

Combining the result from [5] 4.8.1.23 and [Rob] 2.10 (2.50) and 2.25 (2.115), **Sp** can be identified as a homotopy limit of the tower in $\mathbf{Cat}_\infty$

$$... \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

Which is also the homotopy colimit of

$$\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} ...$$

with some modification called "Ind". This is because the later homotopy colimit fails to encode the negative connectivity of the spectra (it would give things bounded below), and the Ind construction will add that back. See detailed Ind construction in [4] 5.3.5.

There is a model category version of this statement, see Hovey's SPECTRA AND SYMMETRIC SPECTRA IN GENERAL MODEL CATEGORIES Theorem 9.3 (or 10.3).

Therefore we have $Ind(Stab_{S^1}(\mathcal{S}_*)) \simeq \mathbf{Sp}$. Replacing $\mathcal{S}_*$ with $\mathcal{S}_*^G$ we obtain the ∞-category of naïve G-spectra $G\mathbf{Sp}$.

Remark 5.6. The author was suggested that if inverting the regular representation sphere S^ρ instead of S^1 in $\mathcal{S}_*^G$, we would obtain the ∞-category of genuine G-spectra. There are a few points to check:

- The (co)limits we are considering are filtered;
- S^ρ is a symmetric object in $\mathcal{S}_*^G$.

Checking these are good exercises at least for myself. The author appreciate all the comments and suggestions.

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[SHGH] J. F. Adams. Stable Homotopy and Generalised Homology.

[Ad] J. F. Adams. Prerequisites (on equivariant stable homotopy) for Carlsson's lecture.

[Ati] M. F. Atiyah. K-theory and reality.

[HILS] F. R. Cohen, T. J. Lada, J. P. May. The Homology of Iterated Loop Spaces.

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