NOTE FOR INFTY CATS

LI, ANG

1. May 7th 2018

(I'm not sure if Nat said something else at the beginning cause I'm late, sorry:) Come to infinity category (infty cat), our primary goal is to have efficient ways to encode coherence. And because infty cat is something falls in btw 1-cat and groupoid (gpd), the technical tool we are using is simplicial sets (sset).

Definition 1.1. A **sset** is a functor (ftr) from the simplex (splx) cat $\mathbf{\Delta^{op}}$ to **Set** equipped with the standard homomorphisms (homo) of presheaves.

(**More useful**) It has the form: $X_0 \leftarrow X_1 \overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}} X_2 ...$

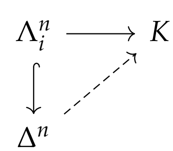
Suppose $\mathscr{C}$ is a small category. $\mathscr{C}$ can be encoded as a simplicial set: $\mathcal{N}(\mathscr{C})$= **nerve** of $\mathscr{C}$. It is defined as follows

$$\mathcal{N}(\mathscr{C})_n = \{strings\ of\ n\ composable\ morphisms\ in\ \mathscr{C}\} = \{X_0 \to X_1 \to ... \to X_n\} = \mathrm{Fun}([n], \mathscr{C}).$$

where $[n] = \{0 \to 1 \to ... \to n\}$. $\mathcal{N}(\mathscr{C})$ remembers the category $\mathscr{C}$: the nerve functor $\mathcal{N}$ is fully faithful. This motivates the definition of an ∞-category (or a quasi-category).

Denote $\Delta^n = \mathcal{N}([n])$, then the n-th level of a sset X satisfies $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X_.) \cong X_n$

Definition 1.2. A simplicial set K is called a **Kan complex** (cplx) if for all n and $0 \leq i \leq n$, any map $\Lambda^n_i \to K$ extends to a map $\Delta^n \to K$.



Remark 1.3. Note that the extension maps do not need to be unique!

Example 1.4. We have taking singular homology ftr: **Top** $\xrightarrow{Sing}$ **sSet**, $Sing_\bullet X$ is a Kan cplx.

Example 1.5. Our nerve ftr: **Cat** $\xrightarrow{\mathcal{N}}$ **sSet**, remember that $\mathcal{N}$ is fully faithful.

A sset X is coming from a cat (is a nerve of some cat) if it has unique lifting wrt **inner horns**, in

Date: Summer 2018.

other words, if for all n and $0 < i < n$, any map $\Lambda^n_i \to X$ extends uniquely to a map $\Delta^n \to X$.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \overset{\exists!}{\nearrow} & \\ \Delta^n & & \end{array}$$

Remark 1.6. [Lurie, Rmk 1.1.2.3] The only intersections known of Kan cplxes and nerve of cats are gpds. (I checked the book again and a little confused about this now......)

Definition 1.7. An ∞-cat (quasicat) is a sset X s.t. for all n and $0 < i < n$, any map $\Lambda^n_i \to X$ admits an extension to a map $\Delta^n \to X$.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Denote $q\mathbf{Cat}$ to be the category of ∞-cats.

The idea is: want the weakest structure with an associated htpy cat and representatives for each mor in the htpy cat.

What about the word "coherence"? How does coherence build in?

Example 1.8. Let $\mathscr{C}$ be an ∞-cat and $\mathscr{D}$ be a 1-cat, then

(1) Maps from $\mathscr{C}$ to $\mathscr{D}$ factor through $h\mathscr{C}$.

$$\begin{array}{ccc} & h\mathscr{C} & \\ \nearrow & & \searrow \\ \mathscr{C} & \xrightarrow{\quad f \quad} & \mathscr{D} \end{array}$$

.

(2) There is an adjuction btw 1-cat and ∞-cat:

$$q\mathbf{Cat} \underset{\mathcal{N}}{\overset{h}{\rightleftarrows}} \mathbf{Cat}$$

given by the nerve ftr and taking htpy.

(3) For a space X, the map from X to $\mathscr{D}$ factor through its fundamental gpd $\Pi_1 X$.

Example 1.9. (1) On the other side, a map from $\mathscr{D}$ to $\mathscr{C}$ is a map of ssets $\mathcal{N}(\mathscr{D}) \to \mathscr{C}$. So for $D_0 \to D_1 \to D_2$ in $\mathscr{D}$, we choose a 2-splx in $\mathscr{C}$ that witnesses that these are composable.

(2) If G is a fin gp, a map $* // G \xrightarrow{f} \mathscr{C}$ gives an obj in $\mathscr{C}$ equipped with a htpy coherent action of G, i.e. need to choose $f(g_1) \cdot f(g_2) \xrightarrow{\simeq} f(g_1 g_2)$. (Here $\simeq$ means equivalence, i.e. have a 2-splx such that the boundary is at the right place). Note that We have no notion of a strict action!

Now we have more thrills coming up:

Example 1.10. A space X and a ftr from $X \to \amalg BGL_n(\mathbb{C}) \subseteq \mathbf{Vect}_{\mathbb{C}} \in \mathbf{Cat}_{\mathbf{Top}}$(cats enriched over **Top**), has the same data of vector bdles by Grothendieck construction.
Replace **Vect** with **Spec**, get parametrized spectrum(?)(I don't know what is this)

Example 1.11. If $\mathscr{C}^{\otimes}$ is a symm mon ∞-cat, then $CAlg(\mathscr{C}^{\otimes})$ = (lax ftrs) $\mathrm{Fun}_{lax}(*^{\otimes}, \mathscr{C}^{\otimes})$ = E_∞-ring in $\mathscr{C}$
lax ftr f satisfies $f(c) \otimes f(d) \to f(c \otimes d)$. For example, $\mathbf{AbGp} \to \mathbf{Set}$, $A \times B \mapsto A \otimes B$.

Proposition 1.12. *(Luries tensor product)*
$\mathscr{C}$, $\mathscr{D}$ presentable ∞-cats, a colim preserving in each var ftr: $\mathscr{C} \times \mathscr{D} \xrightarrow{f} E$ factor through $\mathscr{C} \otimes \mathscr{D}$ uniquely and that functor is colim preserving.

$$\begin{array}{ccccc} & & \mathscr{C} \otimes \mathscr{D} & & \\ & \nearrow & & \searrow^{\exists!} & \\ \mathscr{C} \times \mathscr{D} & & \xrightarrow{\quad f \quad} & & E \end{array}$$

This puts a symm mon str on $q\mathbf{Cat}^{pr,L}$(colim preserving maps are left adjoints), The infty cat of Spaces is the unit. Add stable then the infty cat of spectra is the unit.(?)(I'm not sure I understand the last two sentences)

2. May 10th 2018

We mainly follows [Moritz, page 8-11].

Definition 2.1. A **Kan fibration** is a patten $K \to L$ in **sSet** such that for $0 \leq i \leq n$, we have the lifting

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

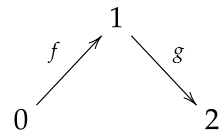
$K \to L$ is a **Kan equivalence** if $|K| \to |L|$ is a weak equivalence in **Top**.

Remark 2.2. If the condition in previous defn changes to $0 \leq i < n$, we call $K \to L$ **left fibration**. If changes to $0 < i \leq n$, we call $K \to L$ **right fibration**.

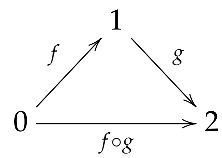
For an ∞-cat $\mathscr{C}$, we want to understand its htpy cat. Our first reasonable thing is to define the set $\mathrm{Hom}_{\mathscr{C}}(x, y)$:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(x,y) & \longrightarrow & \mathscr{C}_1 \\ \big\downarrow & & \big\downarrow d_0 \mid d_1 \\ * & \xrightarrow{(x,y)} & \mathscr{C}_0 \times \mathscr{C}_0 \end{array}$$

How do we make composition concretely defined?
Candidate for comp: If have

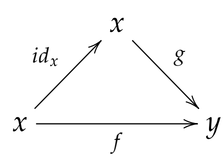


we will have σ as a wittness of htpy to fill in the triangle:



Proposition 2.3. *All candidates are homotopic.*

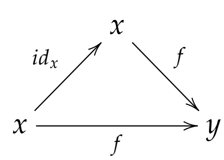
Definition 2.4. [Moritz, defn 1.11] Two morphisms $f, g : x \to y$ in an ∞-cat $\mathscr{C}$ are homotopic if there exists a two splx $\sigma : \Delta^2 \to \mathscr{C}$ such that $\partial\sigma = (g, f, id_x)$:



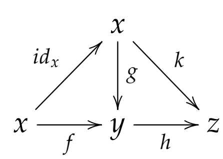
Such σ is called a homotopy btw f and g.

Lemma 2.5. *$f \simeq g$ gives an equivalence relation on* $\mathrm{Hom}_{\mathscr{C}}(x, y)$.

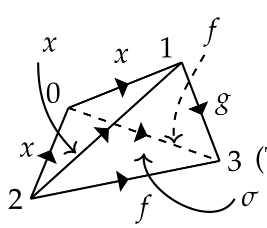
Proof.
- reflection:



- compose: (just put together two triangles)



- symmetric:



The Λ^3_2-horn $(\sigma, id_f, -, id_x)$

(This is the most advanced diag I've stolen from Bert, so excited!)

Where σ is the homotopy from f to g. This horn can be filled in, for we have lifting for $\Lambda^3_2 \to \mathscr{C}$.

■

Since we have the notion of homotopy, we can define the homotopy category of an ∞-category. This gives a ftr from **Cat** to **Cat**$_\infty$ left adjoint to the nerve ftr.

Corollary 2.6. *The htpy cat* $\mathbf{Ho}(\mathscr{C})$ *defined as*

- *obj:* $\mathscr{C}_0$
- *mor:* $\mathrm{Hom}_{\mathscr{C}}(x,y)/\sim$

is a 1-cat.

We have known about the hom set of a ∞-cat, but how to make it a sset? And even more, the mapping space? First, this suffices to make out a sset, where compositions are determined up to contractible choices.

Definition 2.7. Let $X, Y \in \mathbf{sSet}$. Let $i : \Lambda_1^2 \to \Delta^2$ to be the inclusion, denote $\mathrm{Map}(-,-) : \mathbf{sSet}^{op} \times \mathbf{sSet} \to \mathbf{sSet}$ the simplicial mapping space ftr, then define

$$\mathrm{Map}(X,Y)_\bullet = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^\bullet \times X, Y)$$

Under this notion $\mathrm{Map}(X,Y)$ is an ∞-cat if $Y \in q\mathbf{Cat}$.

And starting from there, we can define $\mathrm{Map}_{\mathscr{C}}(x,y)$ to be the pullback square:

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(x,y) & \longrightarrow & \mathrm{Map}(\Delta^1,\mathscr{C}) \\ \downarrow & & \downarrow \\ * & \xrightarrow{(x,y)} & \mathscr{C}\times\mathscr{C} \end{array}$$

We can think this space as "prisms", it is easy to generalize to horizontal composition:

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(x,y,z) & \longrightarrow & \mathrm{Map}(\Delta^2,\mathscr{C}) \\ \downarrow & & \downarrow{\scriptstyle vertices} \\ * & \xrightarrow{(x,y,z)} & \mathscr{C}\times\mathscr{C}\times\mathscr{C} \end{array}$$

So $\mathrm{Map}_{\mathscr{C}}(x,y,z) \to \mathrm{Map}_{\mathscr{C}}(x,y) \times \mathrm{Map}_{\mathscr{C}}(y,z)$ is a trivial Kan fibration (trivial means weak equivalence).

We have two other notions of mapping space of an ∞-cat. First notice that this notion $\mathrm{Map}_{\mathscr{C}}(x,y)$ is indeed a ∞-cat, but doesn't have to be a Kan cplx (I guess), which is bad because in order to develop topological invariant we need somehow a Kan cplx restriction on mapping space, so the htpy type is determined. How to make it a Kan cplx?

Definition 2.8. $\mathrm{Map}^R_{\mathscr{C}}(x,y)$ is defined to be the simplicial set whose n-th layer is n-morphisms from x to y, i.e. a map of simplicial sets

$$f : \Delta^{n+1} \to \mathscr{C}$$

such that $f|_{\Delta\{0,\dots,n\}} = x$ and $f|_{\Delta\{n+1\}} = y$.

Similarly we could define $\mathrm{Map}^L_{\mathscr{C}}(x,y)$, and they are both Kan cplxes.

Here is some guiding principles:

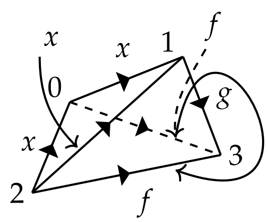
(1) There is a way to compose arrows, but the space of choices is contractible. For instance, 2-horns shows non-emptiness, 3-horns shows connectiveness, and higher horns give higher connectiveness.

To build the space of choices: this is a pullback square

$$\begin{array}{ccc} F_\lambda & \longrightarrow & \mathrm{Map}(\Delta^2, \mathscr{C}) \\ \downarrow & & \downarrow \simeq \\ \Delta^0 & \longrightarrow & \mathrm{Map}(\Lambda^2_1, \mathscr{C}) \end{array}$$

The right side is a weak equivalence thus the space is contractible.

(2) Higher maps are invertible.



the comp is htpc to id_f

Or we can say if we have lifts with respect to outer horns, that brings inverses.

Proposition 2.9. *An ∞-cat $\mathscr{C}$ with outer horn lifts iff $h\mathscr{C}$ is a groupoid, iff $\mathscr{C}$ is a Kan cplx.*

Now we move to [Moritz, section 2].

First we need to introduce "join", the join $A \star B$ of A and B is the thing to make both $A \hookrightarrow A \star B$ and $B \hookrightarrow A \star B$ fully faithful included. And the nerve ftr is compatible with join construction, i.e. there is a natural iso

$$\mathcal{N}(A) \star \mathcal{N}(B) \to \mathcal{N}(A \star B)$$

(See join for 1-cat in [Lurie, section 1.2.8]) Now we define join for ssets.

Definition 2.10. Let $K, L \in$ **sSet**, the join $K \star L$ is the simplicial set such that the n-th layer is defined by

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j \ , \ n \geq 0$$

Proposition 2.11. *The ftrs $K \star -$ and $- \star L$ preserve colimits.*

Example 2.12.

(1) For the standard simplices we have $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}$, for $i, j \geq 0$. The iso is compatible with the inclusions.
(2) Let $K \in$ **sSet**, the **right cone** or **cocone** $K^{\triangleright} = K \star \Delta^0$, and the **left cone** or **cone** $K^{\triangleleft} = \Delta^0 \star K$.

3. May 14th 2018

We are heading to limit and colimit. Before that, we want to talk about equivalence of ∞-cats. Our first version is categorical equivalence.

Definition 3.1. [Lurie, Defn 1.1.5.14] A ftr btw ∞-cats is a map btw ssets $F : \mathscr{C} \to \mathscr{D}$ s.t. F has an inverse $G : \mathscr{D} \to \mathscr{C}$ and a homotopy $H : \Delta^1 \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})$ where $H|_0 = g \circ f$ and $H|_1 = id$, invertible in $h\,\mathrm{Fun}(\mathscr{C}, \mathscr{C})$. (Same as saying F induces an equivalence on $\mathcal{H}$-enriched cat, where $\mathcal{H}$ is the htpy cat of good spaces $\mathcal{CG}$)

Another version is **Dwyer-Kan equivalence**, which is "fully faithful" plus "essentially surjective" generalized to the homotopy cat setting.

Definition 3.2. A ftr btw ∞-cats $F : \mathscr{C} \to \mathscr{D}$ is a **Dwyer-Kan equivalence** if

- $\mathrm{Map}_{\mathscr{C}}(x, y) \to \mathrm{Map}_{\mathscr{D}}(F(x), F(y))$ is a weak-equivalence for any $x, y \in \mathscr{C}_0$.
- the induced map $hF : h\mathscr{C} \to h\mathscr{D}$ is essentially surjective.

And continue on our discussion about the join of ∞-cats, we have:

Proposition 3.3. [Moritz, Prop 2.15]

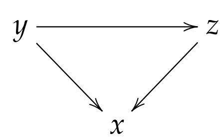
- *If $\mathscr{C}$ and $\mathscr{D}$ are ∞-cats, then the join $\mathscr{C} \star \mathscr{D}$ is an ∞-cat.*
- *If $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{D} \to \mathscr{D}'$ are equivalences of ∞-cats, then also the induced map $F \star G : \mathscr{C} \star \mathscr{D} \to \mathscr{C}' \star \mathscr{D}'$*

To get colimits and limits well situated, we need to talk about **slice category**, which is a generalization of overcategory and undercategory wrt a single object.

In 1-cat setting:

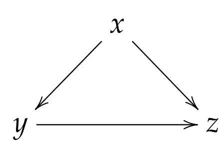
Definition 3.4. Given a cat B and an obj $x \in B$, the overcategory $B_{/x}$ has

- obj: morphisms in B target at x: $y \to x$
- mor: comm triangles:



Similarly, the undercategory $B_{x/}$ has

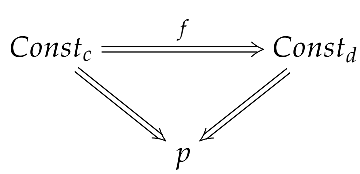
- obj: morphisms in B coming out from x: $x \to y$
- mor: comm triangles:



We can think the object $x \in B$ as a ftr from a single point diag to B. More generally, for a ftr $p : A \to B$ where A is a diag, we can form the **slice category** $B_{/p}$. The objects in $B_{/p}$ are cones on p, i.e. an object $b \in B$ together with a natural transformation from the const ftr on b to p:

$$Const_b \Rightarrow p$$

The morphisms are morphisms $f : c \to d$ in B such that it is compatible with natural transformations:



The slice construction and the join construction (still 1-cat) satisfies a universal property (an adjunction): For every cat C,

$$\mathrm{Fun}(C, B_{/p}) \cong \mathrm{Fun}_p(C \star A, B) \cong \hom_{\mathbf{Cat}_{A/}}(A \to C \star A, A \xrightarrow{p} B)$$

where the later two denote all ftrs $C \star A \to B$ such that the triangle commutes:

$$\begin{array}{ccc} & A & \\ {}^{i}\swarrow & & \searrow^{p} \\ C \star A & \longrightarrow & B \end{array}$$

So in ∞-cat setting, we want to have similar slice construction and universal property wrt the join construction of ssets.

Proposition 3.5. [Moritz, Prop 2.17] *Let $p : L \to \mathscr{C}$ be a map of ssets and $\mathscr{C}$ an ∞-cat, there is an ∞-cat $\mathscr{C}_{/p}$ characterized by the following univ prop: For each $K \in$ **sSet**,*

$$\hom_{\mathbf{sSet}}(K, \mathscr{C}_{/p}) \cong \hom_{\mathbf{sSet}_{L/}}(L \to K \star L, L \to \mathscr{C})$$

We call the ∞-cat $\mathscr{C}_{/p}$ the **∞-cat of cones** *on p.*

4. May 22th 2018

We want to talk about colim and lim, the idea is to compare maps out of (co)lim to (co)lim of mapping space. Here we define final obj:

Definition 4.1. [Moritz, Defn 2.22] An obj x in an ∞-cat $\mathscr{C}$ is a **final obj** if the canonical map $\mathscr{C}_{/x} \to \mathscr{C}$ is an **acyclic fibration** of ssets.
An acyclic fibration is also called a trivial fibration, meaning for every n integer, we have lifting property:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathscr{C}_{/x} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathscr{C} \end{array}$$

And we have several equivalent conditions:

Proposition 4.2. [Moritz, Prop 2.23] *TFAE:*

(1) *The obj x is final.*
(2) *The mapping spaces $\mathrm{Map}_{\mathscr{C}}(x', x)$ are acyclic Kan cplxes for all $x' \in \mathscr{C}$.*
(3) *Every simplicial sphere $\alpha : \partial\Delta^n \to \mathscr{C}$ such that $\alpha(n) = x$ can be filled to an entire n-splx $\Delta^n \to \mathscr{C}$.*

Let's try to see how (1) and (3) are equivalent.

Proof. There is a shift of index happening, the square in previous defn should corresponds to

$$\begin{array}{ccc} \partial\Delta^{n+1} & \xrightarrow{\alpha} & \mathscr{C} \\ & \searrow & \nearrow \\ & \Delta^{n+1} & \end{array}$$

So let's illuminate some easy case:

- (n=0) for (1) we have

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathscr{C}_{/x} \\ \downarrow & \nearrow_{f} & \downarrow \\ * & \longrightarrow & \mathscr{C} \end{array}$$

f gives a map from $* \star * \to \mathscr{C}$ through the "slice join" adjunction, sending the second point to x. Therefore if consider (3)

$$\begin{array}{ccc} *_a \; *_b & \longrightarrow & \mathscr{C} \\ & \searrow & \nearrow \\ & *_a \to *_b & \end{array}$$

we are able to fill in this edge because the image of f is in $\mathscr{C}_{/x}$ thus it is an edge in $\mathscr{C}$.

- (n=1)

$$\begin{array}{ccc} *_a \; *_b & \longrightarrow & \mathscr{C}_{/x} \\ \big\downarrow & \overset{f}{\nearrow} & \big\downarrow \\ *_a \to *_b & \longrightarrow & \mathscr{C} \end{array}$$

This gives a map from $(*_a \to *_b) \star * \to \mathscr{C}$, or we can think of the image of f is a mor in $\mathscr{C}_{/x}$. Correspondingly in (3) we have

$$\begin{array}{ccc} a & \longrightarrow & b \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

and

$$\begin{array}{ccc} f(a) & \longrightarrow & f(b) \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

both map into $\mathscr{C}$ with $*$ maps to x. And we are able to fill in both.

■

From these process we will have a feeling that the space of terminal objs is contractible.

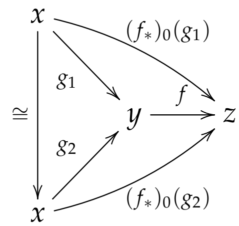
Another thing could say about how limit behaves. In 1-cat setting

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim_{\mathscr{C}} Y_\alpha) \xrightarrow{\cong} \lim_{Kan} \mathrm{Hom}_{\mathscr{C}}(X, Y_\alpha)$$

but in ∞-cat setting we probably need to add conditions.
In general, a map from $f : \Delta^1 \to \mathscr{C}$ connecting $y \to z$ will give rise to $\mathrm{Map}_{\mathscr{C}}(x, y) \xrightarrow{f_*} \mathrm{Map}_{\mathscr{C}}(x, z)$.
In particular, on 0-splx: for every $x \xrightarrow{g} y$ we get a horn Λ^2_1

$$\begin{array}{ccccc} & & y & & \\ & \overset{g}{\nearrow} & & \overset{f}{\searrow} & \\ x & & & & z \end{array}$$

And there exists a choice of $x \to z$ for every g, denote $(f_*)_0(g)$.
For 1-splx, for every $x \xrightarrow{g_1} y$ and $x \xrightarrow{g_2} y$, we have a horn Λ^3_2



We have choices to fill in the missing face as well. We try to get to (co)limit using this argument.

Definition 4.3. [Moritz, Defn 2.27] Let K be a sset and $\mathscr{C}$ an ∞-cat, a **colimit** of a diag $p : K \to \mathscr{C}$ is an initial obj in $\mathscr{C}_{p/}$. $\mathscr{C}$ is **cocomplete** if it admits colimits of all small diag.
Dually we can define limits and complete ∞-cats.

Remark 4.4. The nerve ftr $\mathcal{N} : \mathbf{Cat} \to \mathbf{sSet}$ is compatible with (co)limits.

Now we would like to talk about presentable ∞-cats (No I'm sure we don't really want to talk about this), thus we will have adjoint ftr thm (a ftr is left adjoint iff it preserves colimit, and similar version for right adjoint).
But, it has more better properties that I don't really understand yet, for instance, in [Moritz] it says that **a more precise relation between ∞-cats and model cats are indicated**, which I'm really curious. Anyway, Nat said something that those two approaches will merge in less than 10 years and I totally believe that.

OK now, do I really want to talk about presentable? No. But let's have a better but still informal statement of the Adjoint Functor Theorem:

Theorem 4.5. *Let $\mathscr{C}$, $\mathscr{D}$ cats with $\mathscr{C}$ having limits. $\exists$ ftr $F : \mathscr{C} \to \mathscr{D}$ preserves all colimit, satisfying blah......(don't laugh please)*
Then $\exists$ a right adjoint $G : \mathscr{D} \to \mathscr{C}$ of F.

So we want to dig out the analog environment such that this thm works, in the ∞-cat land.

Definition 4.6. A **presentable ∞-cat** is a ∞-cat that is cocomplete and accessible.
$\mathscr{C}$ is **accessible** if it is κ-accessible for some regular cardinal κ. (I know this sounds like a "blah..." but try to think this as a smallness condition)

This is not the worst (it is never...), we also need restrictions on our diag cats.

Definition 4.7. A diag cat $\mathscr{D}$ is **filtered** if

- nonempty
- for any two objects $a, b \in \mathscr{D}$, there exists an obj $c \in \mathscr{D}$ such that we have arrows $a \to c$ and $b \to c$
- always have coequalizers: for any $a \underset{g}{\overset{f}{\rightrightarrows}} b$, there exists $b \xrightarrow{h} c$ such that $hf = hg$

Equivalently, we can say for any fin diag X and a map $X \to \mathscr{D}$, there exists a map from the cocone of X: $X^{\triangleright} \to \mathscr{D}$.
$\mathscr{D}$ is κ-filtered if for any X with $|X| < \kappa$ and any $X \to \mathscr{D}$, there exists $X^{\triangleright} \to \mathscr{D}$.

Then there are some description of κ-accessible ∞-cats in terms of κ-filtered colim, and whatever $Ind_{\kappa}(\mathscr{C})$ that makes Bert want to throw up......I gave up.

5. May 30th 2018

Let's assume we have presentability after a terrible discussion, thus we have celebrated adj ftr thm at hand. Now we would like to generalize the Yoneda Lemma and the Yoneda imbedding Thm to ∞ setting.
First in 1-cat, we have

Lemma 5.1. *(Yoneda)*
*Let $\mathscr{C} \in$ **Cat**, F a ftr from **Cat** to **Set**, then $\exists x \in \mathscr{C}$ s.t.*

$$Nat(\mathrm{Hom}_{\mathscr{C}}(-,x), F) \cong F(x)$$

Theorem 5.2. *(Yoneda imbedding)*

$$\mathscr{C} \xrightarrow{fully\ faithful} \mathrm{Fun}(\mathscr{C}^{op}, \mathbf{Set})$$

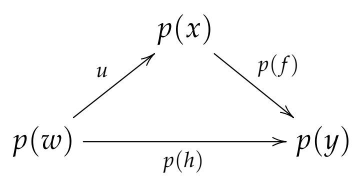
by $x \mapsto \mathrm{Hom}_{\mathscr{C}}(-,x)$.

In ∞-cat setting, we expect something "$\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{op}, \mathcal{S})$" where $\mathcal{S}$ is the **∞-cat of spaces**, technically in [Lurie, 1.2.16], $\mathcal{S} = \mathcal{N}_s(\mathbf{Kan})$, where $\mathcal{N}_s$ is the **simplicial nerve**. (it plays the roll of **Set** for ∞-cats)
But the reasonable assignment $x \mapsto \mathrm{Map}_{\mathscr{C}}(-,x)$, the target is not functorial. Recall we only have zig-zag of maps:

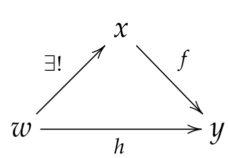
$$\mathrm{Map}_{\mathscr{C}}(x,y) \times \mathrm{Map}_{\mathscr{C}}(y,z) \overset{\simeq}{\leftarrow} \mathrm{Map}_{\mathscr{C}}(x,y,z) \to \mathrm{Map}_{\mathscr{C}}(x,z)$$

In order to solve this, we need (Cartesian) fibrations.

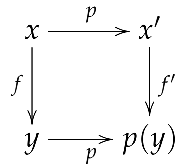
Definition 5.3. (in 1-cat setting)
For a ftr $X \xrightarrow{p} S$, and $x \xrightarrow{f} y \in X$, we say f is p-**Cartesian** if for any u and any $w \xrightarrow{h} y$ such that the following commutes



we have



also commutes.
Then we can define $p : X \to S$ is a **Cartesian fibration** if for any $x' \xrightarrow{f'} p(y) \in S$, we have a p-Cartesian $f : x \to y$ in X such that $p(f) = f'$



Among Cartesian fibrations, there are spetial ones called discrete fibration:

Example 5.4. $X \xrightarrow{p} S$ is called a discrete fibration if

- p is a Cartesian fibration
- fibers (pullbacks) over a pt is a set, i.e.

$$\begin{array}{ccc} \mathbf{Set} \ni X_s & \longrightarrow & X \\ \downarrow & & \downarrow p \\ * & \xrightarrow[s]{} & S \end{array}$$

If we look at the **Grothendieck construction** on 1-cat, for a ftr $F : \mathscr{C}^{op} \to \mathbf{Cat}$, define $\int_{\mathscr{C}} F$:

- obj: (x, A) where $x \in \mathscr{C}$ and $A \in F(x)$
- mor: $(x, A) \xrightarrow{(f,\varphi)} (y, B)$ where $x \xrightarrow{f} y \in \mathscr{C}$ and $A \xrightarrow{\varphi} f^*(B) \in F(x)$

And we have the following theorem:

Theorem 5.5. *We have categorical equivalences*

$$\mathrm{Fun}(\mathscr{C}^{op}, \mathbf{Cat}) \xrightarrow{\int_{\mathscr{C}}(-)} \{Cartesian\ fibrations\ over\ \mathscr{C}\}$$

$$\mathrm{Fun}(\mathscr{C}^{op}, \mathbf{Set}) \xrightarrow{\int_{\mathscr{C}}(-)} \{discrete\ fibrations\ over\ \mathscr{C}\}$$

Example 5.6. If we consider the assignment $\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{op}, \mathbf{Set})$ by $x \mapsto \mathrm{Hom}_{\mathscr{C}}(-, x)$, then the Grothendieck construction of $\mathrm{Hom}_{\mathscr{C}}(-, x)$ is the slice construction $\mathscr{C}_{/x}$. So we know the projection $\mathscr{C}_{/x} \to \mathscr{C}$ is a discrete fibration and we can think of slice cats as representable.

In other words, we can restate Yoneda Embedding thm and Yoneda Lemma via slice construction.

Lemma 5.7. *(Yoneda)*
*Let $\mathscr{C}, \mathscr{D} \in$ **Cat**, $x \in \mathscr{C}$, then*

$$\mathrm{Hom}_{dFib(\mathscr{C})}(\mathscr{C}_{/x}, \mathscr{D})_{(id_x)_*} \to \mathrm{Hom}_{dFib(\mathscr{C})}(*, \mathscr{D}) = \mathscr{D}_x$$

is an equivalence of cats, where $dFib(\mathscr{C})$ indicates discrete fibrations in $\mathscr{C}$.

Theorem 5.8. *(Yoneda imbedding)*

$$\mathscr{C} \xrightarrow{fully\ faithful} dFib(\mathscr{C})$$

by $x \mapsto (\mathscr{C}_{/x} \to \mathscr{C})$, the projection.

What about fibrations in **sSet**? First we define p-Cartesian arrow.

Definition 5.9. Let $p : X \to S \in$ **sSet** and $x \xrightarrow{f} y \in X$, we say f is p**-Cartesian** if (f is the last edge)

$$\begin{array}{ccc} \Delta^{\{n-1.n\}} & & \\ \downarrow & \searrow^{f} & \\ \Lambda^n_n & \longrightarrow & X \\ \downarrow & \nearrow_{\exists !} & \downarrow \\ \Delta^n & \longrightarrow & S \end{array}$$

Equivalently, if

$$X_{/f} \to X_{/y} \underset{S_{/p(y)}}{\times} S_{/p(f)}$$

is a trivial Kan fib.

The defination of Cartesian fibration for sset is the same as 1-cat setting (5.3). And right fibration is the parallel version for discrete fibration in 1-cat setting. i.e.

right fibration = (recall 2.2)

- cartesian fibration
- fibres over pt are Kan cplxes

So let's try to state the Yoneda lemma and embedding thm for ∞-cats, or more concretly, for ssets.

Lemma 5.10. *(Yoneda)*
*Let $\mathscr{C}, \mathscr{D} \in$ **sSet**, $x \in \mathscr{C}$, then*

$$\mathrm{Hom}_{Fib^R(\mathscr{C})}(\mathscr{C}_{/x}, \mathscr{D}) \to \mathrm{Hom}_{\mathscr{C}}(\Delta^0, \mathscr{D}) = \mathscr{D}_x$$

is an equivalence of Kan cplxes, where $Fib^R(\mathscr{C})$ indicates right fibrations in $\mathscr{C}$.

Theorem 5.11. *(Yoneda imbedding)*

$$\mathscr{C} \xrightarrow{fully\ faithful} Fib^R(\mathscr{C})$$

by $x \mapsto \mathscr{C}_{/x}$.

There are more to say about $Fib^R(\mathscr{C})$, if we take the htpy of the simplicial nerve (5) of it, we get $P'(\mathscr{C})$, it is closely related to $P(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}^{op}, \mathcal{S})$, the presheaves on $\mathscr{C}$. There are some zig-zag weak equivalence btw them.

In 1-cat setting, suppose we know the Grothendieck construction on **Cat** $\xrightarrow{id}$ **Cat** to be $\int_{\mathbf{Cat}} id = \mathbf{Cat}_*$, then we have pullback on cats

$$\begin{array}{ccc} \int_{\mathscr{C}} F & \longrightarrow & (\int_{\widehat{\mathbf{Cat}}} id)^{op} \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow[F]{} & (\widehat{\mathbf{Cat}})^{op} \end{array}$$

(I forget what does the hat mean)
This is saying everything is given by the identity, we denote the Grothendieck construction of **Cat** $\xrightarrow{id}$ **Cat** the **universal category fibration**.

Similarly, we have universal fibrations to classify right fibrations. First, $\int_{q\mathbf{Cat}} id \to q\mathbf{Cat}$ characterizes all ∞-cats,

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathcal{Z} = \int_{q\mathbf{Cat}} id \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[\mathscr{C}]{} & q\mathbf{Cat} \end{array}$$

the univ right fib is given by the left column:

$$\begin{array}{ccc} \mathcal{Z}^\circ & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{S}^{op} & \longrightarrow & q\mathbf{Cat} \end{array}$$

where $\mathcal{S}$ is the ∞-cat of spaces (5), and that gives all the right fibs $\widetilde{\mathscr{C}} \to \mathscr{C}$, which are contained in $P'(\mathscr{C})$.

$$\begin{array}{ccc} \widetilde{\mathscr{C}} & \longrightarrow & \mathcal{Z}^\circ \\ \downarrow & & \downarrow \\ \mathscr{C} & \longrightarrow & \mathcal{S}^{op} \end{array}$$

Theorem 5.12. [Moritz, thm 3.16] *Let* $\mathrm{Fun}^L(X, Y)$ *denote the ftrs from* X *to* Y *that is cts wrt colims,* $\mathscr{C} \in$ **sSet** *and* $\mathscr{D} \in q$**Cat**. *We have*

$$\mathrm{Fun}^L(P(\mathscr{C}), \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

is an equivalence of ∞*-cats.*

Corollary 5.13. $\mathcal{S}$ *is freely gen by* Δ^0*, i.e.* $\mathcal{S} = P(\Delta^0)$. *And we have*

$$\mathrm{Fun}^L(\mathcal{S}, \mathscr{D}) \cong \mathscr{D}$$

References

[Moritz] M. Groth. *A Short Course on ∞-Categories.* Available at https://arxiv.org/pdf/1007.2925.pdf.
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