Project 2 - Direction Fields and Solution Curves, Part 2

Due in class on Friday, 26 February 2010. Please staple your project!

These notes basically are from the IODE project and are a modification of those written by R. Laugesen.

1. Qualitative properties of solution curves

For the next project, we need the concept of **periodicity**. A function $g(x)$ is called **periodic** if for some number $P > 0$ one has $g(x) = g(x + P)$ for all $x \in \mathbb{R}$.

The smallest $P > 0$ for which this relation holds is called the **period** of the function g.

Example 1. The function $g(x) = \sin(x)$ is periodic with $P = 2\pi$, since $\sin(x) = \sin(x + 2\pi)$ for all x. Notice that the graph of $\sin x$ repeats itself every 2π units, which we expect from the periodicity.

Example 2. The function $g(x) = \sin(x) + 1$ is periodic with $P = 2\pi$, since $\sin(x) + 1 = \sin(x + 2\pi) + 1$, for all x.

Example 3. The function $g(x)$ defined in a piecewise manner by the rule

$$g(x) = \begin{cases} 1 & -2 \le x < -1, 0 \le x < 1, 2 \le x < 3, \ldots \\ 2 & -1 \le x < 0, 1 \le x < 2, 3 \le x < 4, \ldots \end{cases}$$

is periodic with $P = 2$.

(Exercise: sketch a graph of this function!) This example shows periodic functions do not have to be trigonometric functions.

Example 4. Consider the more complicated function $g(x, y) = 3^{\cos 2x} + y^2$. This function of two variables is periodic in the x-variable with period $P = \pi$, because $g(x, y) = g(x + \pi, y)$ for all x and y, by the previous example. But $g(x, y)$ is not periodic in the y-variable, because no matter what $P > 0$ you try, you can check that $g(x, 0) \neq g(x, P)$.

2. Lab Project

Provide a complete explanation for each of the following problems. A complete explanation must include examples and plots using Iode. An explanation makes reference to the most relevant features of the ODEs and direction fields as discussed in lab one. You should do the same wherever possible. Use Iode to create a few examples of relevant direction fields and solution curves, to help give you ideas. Include these plots with your solution.

Answer the questions in each exercise and provide a complete explanation for each answer.

(1) The solution curves of $y'(x) = f(x, y)$ are all increasing from left to right if f has what property? Give an example of a function f satisfying this condition. Does $f(x, y) = x^2 + y^2 + 1$ satisfy this condition? Use Iode to draw the direction field and plot some solutions.

(2) Suppose $y = y(x)$ solves $y'(x) = f(x)$. If $f(x)$ is periodic then is it true that $y(x)$ is periodic? The solution is given by

$$y(x) = \int_a^x f(s)\ ds,$$

for any constant a. What does it mean for y to be periodic? Explain and give some examples. Consider $f(x) = \cos(2x)$ and $f(x) = \sin(x) + 1$. Plot the direction fields. Solve the two ODEs by hand. What are the properties of $y(x)$? Can you find a general condition on f in order to have y periodic?

(3) The plot of the direction field for $y'(x) = f(x, y)$ shows a vertically repeating pattern such as that obtained for $f(x, y) = x \sin y$. Explain this and give some examples.

(4) Suppose $y(x)$ solves $y'(x) = f(x)$. If $y(x) \to \infty$ as $x \to \infty$, then necessarily $f(x) \to \infty$ as $x \to \infty$. Is this true? Give an example and make some plots using Iode.

(5) Consider $y'(x) = f(y)$, and suppose $f(2) = 0$. What feature do you observe in the direction field along $y = 2$? Give an example and use Iode to draw the direction fields and some solutions.

(6) Find a function $f(y)$ with the property that: for each solution $y(x)$ of $y'(x) = f(y)$, the limiting value $\lim_{x\to\infty} y(x) = 3$ if $y(0) > 0$. Your explanation should consist of a direction field with illustrative solution curves plotted on it.

(7) **HINT:** For problems 5 and 6, think about the logistic equation that we studied in section 2.5.