# Project 3 - Forced Oscillations

**Due in class on Friday, 4 December 2009. Please staple your project!**

These notes basically are from the IODE project and are a modification of those written by V. Zharnitsky and J. T. Tyson.

## 1. Second-order constant coefficient ODEs

Goal of the project is to consider how the solution of forced oscillator ODE depends on the various parameters. We consider ODEs of the form

$$(1.1) \qquad mx''(t) + \gamma x'(t) + kx(t) = F(t),$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator, $\gamma \geq 0$ is the damping coefficient, and $F(t)$ is the external forcing term. We are going to investigate graphically the phenomena of: resonance, beating, transient behavior, and steady state oscillations. These all depend on the relation between $\omega_0$, $\gamma$ and the forcing term $F$.

## 2. Goals of the Project and Getting Started

First, start IODE and click on **Second order linear ODEs** in the main menu. You will get an interface very similar to the one for direction fields which we used in Projects 1 and 2. Go to the right side of the screen and click on **Solution Method**. It is normally set to Euler. Change it to Runge–Kutta. This is a more accurate numerical scheme. The Euler method is far too crude for what we are going to do in this project.

To ensure that you have the correct set up, try entering the following ODE:

$$(2.1) \qquad x'' + x = 0.$$

Notice that when you enter an ODE into the **Enter differential equation** window, you must specify $m, c, k$ and the forcing function $f(t)$ for the ODE:

$$(2.2) \qquad mx''(t) + cx'(t) + kx(t) = f(t).$$

*You must enter some number on each line even if it is zero. Do not leave empty spaces as IODE will not understand that.* Now plot the solution with initial condition $(x(0), x'(0)) = (1, 0)$. Then plot the solution with initial condition $(x(0), x'(0)) = (0, 1)$. You should see on the screen the graphs of $\cos(t)$ and $\sin(t)$ respectively.

You may specify the initial conditions with the mouse and then IODE will plot the solution curves. Simply press down the left mouse button at the desired initial point $(t_0, x(t_0))$, and then drag the mouse at the desired slope a short distance to specify $x'(t_0)$, since you need to specify two conditions at $t_0$. When you release the mouse, IODE will plot the solution passing through this point with this slope. NOTE: If you just press down the left mouse button and release it without dragging, then the initial slope will be taken as one, that is $x'(t_0) = 1$.

## 3. The Project Components

### 3.1. Beating solutions for an undamped, forced oscillator. Set the display parameters to

$$(3.1) \qquad -100 < t < 100, \text{ and} -30 < x < 30.$$

Plot the solution of the ODE:

$$(3.2) \qquad x''(t) + x(t) = \cos(1.1t),$$

with initial condition $(x(0), x'(0)) = (0, 0)$. You should see a beautiful beat pattern!

(1) Graphically estimate the period of the beating (attach your solution plot), which is defined to be twice the distance between the neighboring troughs. Can you explain why it has this value?

   **Hint**: The period of beating $\tau$ equals $2\pi$ divided by the frequency $\omega_b$ of beating: $\tau = 2\pi/\omega_b$. The trig identity

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

   is useful to explain this phenomenon.

(2) Theoretical explanation: What happens to the period of beating as you change the forcing frequency from $\omega = 1.1$ to $\omega = 1.05$? Change the time scale to $-200 < t < 200$ in order to see the beats clearly. Attach a plot. What happens to the amplitude of the beating?

### 3.2. Resonance behavior for an undamped, forced oscillator. Plot the solution of the ODE:

$$(3.3) \qquad x''(t) + x(t) = \cos t,$$

with initial condition $(x(0), x'(0)) = (0, 0)$. Note the forcing frequency $\omega = 1$ is equal to the natural frequency $\omega_0 = 1$ of the system. Make sure your scales are set so that you see an oscillating solution with growing amplitude. The solution should look like the graph of a function of the form $ct \sin t$. Graphically estimate the value of $c$ using the slope of the envelope of the solution. Check your answer as follows.

(1) Use the **Plot arbitrary function** command to plot the amplitude of the envelope curve $x(t) = ct \sin t$. Do the two envelopes agree?
(2) Exactly solve the ODE and find the value of $c$.

Submit these results and plots.

Considering the graphs and solutions which you found in problem 1, in what sense (if any) can you say that the solution to the initial value problem

$$(3.4) \qquad x''(t) + x(t) = \cos(\omega t), \ x(0) = 0; x'(0) = 0,$$

approaches the solution to the initial value problem at resonance $\omega_0 = \omega = 1$ considered here? **Hint:** Think about how the period and amplitude of the beating change as $\omega$ is decreased from 1.1 to 1.05 or as it is increased from 0.5 to 0.8 to 1.

### 3.3. Behavior of a damped, unforced oscillator. We saw in class that the behavior of solutions to

$$(3.5) \qquad mx''(t) + \gamma x'(t) + kx(t) = 0,$$

depend on the sign of the discriminant $\gamma^2 - 4mk$. Consider the ODE:

$$(3.6) \qquad x''(t) + \gamma x'(t) + x(t) = 0,$$

and plot solutions to the initial value problem $(x(0), x'(0)) = (0, 1)$ for $\gamma = 0.5$, for $\gamma = 2$ and for $\gamma = 3$. Describe the observed behavior in each case and relate it to the exact solutions. Why do the solutions in the first and last case look the same?

### 3.4. Transient behavior of a damped, forced oscillator.

The general ODE (1.1) has a general solution having the form

$$(3.7) \qquad x(t) = x_h(t) + x_p(t),$$

where $x_h(t)$ is the most general solution of the associated homogeneous ODE (3.5) and $x_p(t)$ is a particular solution of the nonhomogeneous ODE.

We know that from class and from problem 3 that as long as $\gamma > 0$, we have that the amplitude goes to zero: $\lim_{t\to\infty} x_h(t) = 0$. So for large times, only the particular solution of (1.1) remains.

How large is large? When we do some plots described below, we see an initial behavior, called the *transient behavior*, this is when both $x_h(t)$ and $x_p(t)$ are contributing to the solution, followed by a regular pattern called the *steady-state behavior* that come mostly from $x_p$.

What is $x_p(t)$? Let's take $F(t) = F_0 \cos(\omega t)$. An intelligent guess is that it has the form

$$(3.8) \qquad x_p(t) = A\cos(\omega t) + B\sin(\omega t).$$

We can determine $A$ and $B$ by substituting this $x_p$ into the ODE. This is done in our text on pages 207-209 and discussed there. The result is that the steady-state solution looks like:

$$(3.9) \qquad x_p(t) = \frac{F_0}{\Delta}\cos(\omega t - \delta),$$

where

$$(3.10) \qquad \Delta = [m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}.$$

Note that when $\gamma \neq 0$, this never blows up, even at resonance $\omega = \omega_0$! But it is as big as it can be at resonance. The frequency of the steady-state solution is $\omega$, the driving frequency.

For this example, use display parameters $0 < t < 20$ and $-10 < x < 10$. Plot solutions of the equation

$$(3.11) \qquad x''(t) + x'(t) + x(t) = \cos(0.5t),$$

with a variety of initial conditions. Choose the initial conditions by clicking and dragging the mouse. All initial conditions should be taken near $t = 0$, with $x(0)$ and $x'(0)$ not too large. You should see that solutions with different initial conditions approach the same solution after some time $\tau$. We say **transient** behavior occurs during the time interval $[0, \tau]$. After that time, all the different solutions appear to coincide on the graph and you observe only steady oscillations.

(1) Graphically estimate $\tau$. Attach the plots.
(2) Can you explain why $\tau$ has this value? Give a theoretical explanation.
(3) In this above example, we have $c = 1$. What happens (experimentally) to the transient time $\tau$ as $c$ decreases? Try $c = 0.5$ and $c = 0.25$.
(4) What happens as $c$ increases? Try $c = 2$, $c = 4$ and $c = 8$. Attach some graphs.
(5) Can you explain intuitively the observed dependence of $\tau$ on the dampening coefficient $c$? Can you explain it theoretically?

3.5. **Steady-State Behavior of a damped, forced oscillator at and near resonance.** We now consider a damped, forced oscillator at resonance:

(3.12) $$x''(t) + x'(t) + x(t) = \cos(\omega t), \ x(0) = 0, \ x'(0) = 0,$$

Try varying the forcing frequency $\omega$ below and above the natural frequency $\omega_0$. Try $\omega = 0.2, 0.5$ and $\omega = 1.2, 2$. Finally, look at the resonance case $\omega = \omega_0 = 1$. Attach the graphs.

(1) How does the amplitude $C(\omega)$ change with $\omega$? When is $C(\omega)$ the largest?

(2) How does this compare to the undamped case studied in problem 2?

(3) Can you explain intuitively the observed dependence of the amplitude $C(\omega)$ on the forcing frequency ? Can you explain it theoretically? **Hint:** In order to get an accurate estimate for $C(\omega)$, it may be useful to restrict the viewing window in the $x$-variable. Try the display parameters $0 < t < 50$ and $-1.5 < x < 1.5$ for $\omega > \omega_0$ and $-1 < x < 1$ for $\omega > \omega_0$.