MA214 Section 003 Spring 2010
Practice Test #1
24 February 2010

REVIEW SESSION: Wednesday, 24 February, 4PM–5PM, CB 313

MATERIAL: Chapter 1; Chapter 2, Sections 1, 2, 3, 5, 7; Chapter 3, Sections 3.1, 3.2, and 3.3 (you do not have to know Abel's formula.)

1. The velocity of a falling body in air satisfies

$$m\frac{dv}{dt} = mg - \gamma v,$$

   where $m > 0$ is the mass of the body, the constant $g > 0$ is the gravitational constant, and $\gamma > 0$ is the drag due to air resistance.

   (a) Find the most general solution to the equation.

   (b) Suppose that $g = 10m/sec/sec$ (an approximation), the body has mass $m = 10kg$, and the drag coefficient is $\gamma = 2kg/sec$. If $v(0) = 0$, find the body's velocity at any time $t > 0$.

   (c) The body is dropped from a height of $300m$. Write the equation for the position $x(t)$, measured positively from $x = 0$.

   (d) How long does it take for the body to hit the ground? Simply write the equation satisfied by the time T.

2. Are the functions $f(x) = \cos(x)$ and $g(x) = \cos(2x)$ linearly independent on $I = [0, 2\pi]$?

3. Solve the ODE:

$$y'' + 6ty' = 6t, \; y'(t = 0) = 1, \; y(t = 0) = 1,$$

   by reduction of order. Let $u(t) = y'(t)$ and find the solution $u(t)$ satisfying the initial condition. Then find the solution $y(t)$. Check your answer.

4. Suppose a sum S_0 is invested at an annual rate of return r compounded continuously. Find the time T it takes for the original sum to double as a function of r. What must r be if the sum doubles in 8 years?

5. Clearly identify the direction field of the ODE: $y' = 4 - 2y$. Sketch the direction field and describe its main features. What is $\lim_{t \to +\infty} y(t)$?

6. Find the most general solution to the ODE

$$y'' + 4y' - (9/4)y = 0.$$

   Be sure to check that you have two linearly independent solutions.

7. A tank initially contains 100 liters of pure water. A stream of polluted water with a concentration of $\gamma = 5$ grams per liter of mercury is poured into the tank at a rate of one liter per minute. It flows out at the same rate. How much mercury is in the tank after $100 \ln 2$ minutes?

8. Find the general solution to the ODE:

$$ty'(t) + 3y(t) = 6t^2.$$

Find the unique solution to the ODE with initial condition $y(1) = 0$.

9. Solve the ODE:

$$y' = \frac{3x^2 - 1}{3 + 2y}.$$

10. The population of bacteria $P(t)$ satisfies the ODE:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P,$$

for $t \geq 0$. The constants $r > 0$ and $K > 0$ represent the growth rate and the saturation level, respectively.

   a. Find the population if the initial population $P(0)$ satisfies $0 < P(0) < K$.

   b. What is $\lim_{t\to\infty} P(t)$ in case (a.)?

11. A room contains 100 cubic meters (m^3) of gas with an initial concentration of $5g/m^3$ of a poisonous gas. Air enters the room at a rate of $20m^3/min$ and has a concentration of $5e^{-t/4}\ g/m^3$ of poison. Air flows out of the room at the same rate. Find the quantity $Q(t)$ of poison in the room at any $t > 0$. Recall that

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out}),$$

and the rate in or out is the flow rate times the concentration.

12. Describe Euler's method for the ODE: $y'(t) = f(t, y)$. Let $f(t, y) = y$. Find the exact solution with the initial condition $y(0) = 1$. Now take a step size $h = 1$ and compute the approximation at $t = 3$. What is the exact value?

13. Find the unique solution to the initial value problem

$$y'' - 3y' + y = 0, \quad y(0) = 0, \quad y'(0) = 5.$$