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(Sections 1.3-1.4)

Concepts

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(Sections 1.3-1.4)

2-dimensional Pictures

In this section, we leap from the one-dimensional number line to the two-dimensional Cartesian Coordinate System. This leap will allow us to see pictures of equations that contain two variables (x and y , for example). We will also see how these pictures can be used to help us approximate solutions of equations that contain only one variable.

Your Personal Review

Undoubtedly, you have seen the Cartesian Coordinate System in previous classes. We will not review the basics of plotting points during lecture, but you should read section 1.3 in your textbook for a quick review. In particular, you need to be able to:

- Locate points on a Cartesian Coordinate System.
- Discuss the four quadrants of a Cartesian Coordinate System.
- Identify the x -axis, the y -axis, and the origin on a Cartesian Coordinate System.

Your Personal Review

Example 1 (A Review of the Cartesian Coordinate System)

Every point on the x -axis has:

- A. an x -coordinate that equals 0.
- B. a y -coordinate that equals 0.
- C. Both (A) and (B).

Graphs of Equations with Two Variables

Definition 2

The graph of an equation in the variables x and y is the set of all points (a, b) such that $x = a$, $y = b$ is a solution to the equation.

- If a point is on the graph of an equation, then the point is a solution of the equation.
- If a point is a solution of an equation, then the point is on the graph of the equation.
- If a point is **not** on the graph of an equation, then the point is **not** a solution of the equation.
- If a point is **not** a solution of an equation, then the point is **not** on the graph of the equation.

Graphs of Equations with Two Variables

Example 3

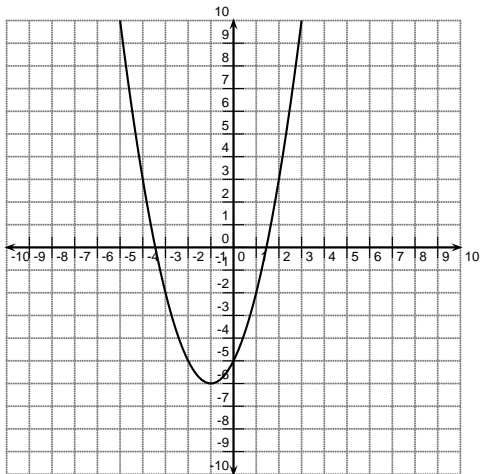
- Is $(2, 3)$ on the graph of $y = 3x + 5$?

- Is $(1, 8)$ on the graph of $y = 3x + 5$?

Graphs of Equations with Two Variables

Example 4

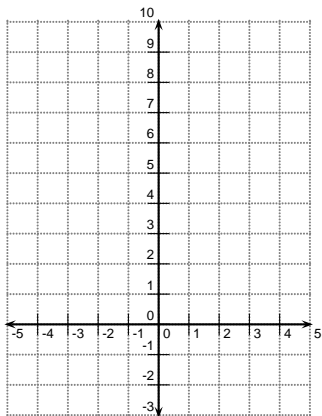
Explain why the graph below is **not** the graph of $y = x^2 + 2x + 3$.



Graphs of Equations with Two Variables

Example 5

Sketch the graph of $y = |x^2 - 4|$ by making a table of values.



Graphs of Equations with Two Variables

Definition 6 (x -intercepts and y -intercepts)

If a graph intersects the x -axis at the point $(a, 0)$, then a is called an x -intercept of the graph.

If a graph intersects the y -axis at the point $(0, b)$, then b is called a y -intercept of the graph.

Notice that the y value equals zero at an x -intercept and the x -value equals zero at a y -intercept.

Graphs of Equations with Two Variables

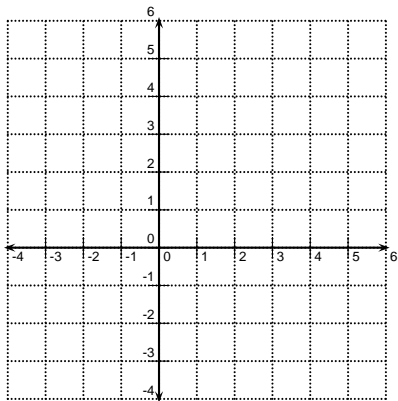
Example 7

Find the intercepts of the graph of $x = y^2 + 2y - 3$

Distance

Example 8

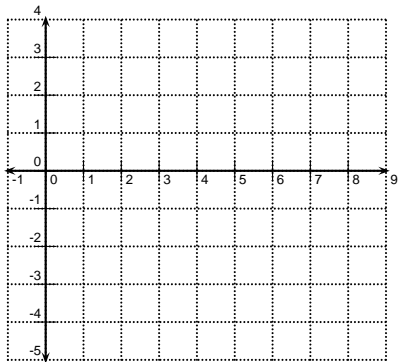
Use the Pythagorean Theorem to find the distance between the points $(4, 5)$ and $(1, -3)$.



Distance

Example 9

Use the Pythagorean Theorem to find the distance between the points $(6, -4)$ and $(b, 3)$.



Distance

Theorem 10 (The Distance Formula)

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The Distance Formula is just the Pythagorean Theorem in disguise.

Distance

Example 11

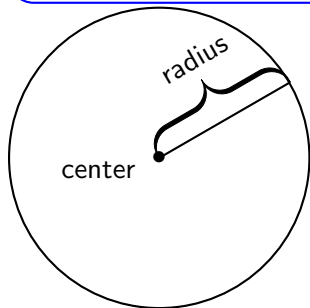
Find the perimeter of the triangle with vertices $A(-2, -5)$, $B(-2, 7)$, and $C(10, 10)$.

Equations of Circles

Recall the definition of a circle.

Definition 12

A **circle** is the set of all points that are a fixed distance r from a specified point called the **center** of the circle. The distance r is called the **radius** of the circle.



Equations of Circles

Since the definition of a circle is based on a distance, we can use the distance formula to find the equation of a circle.

Example 13

Find an equation for the circle with center $(-2, 5)$ and radius 4.

Equations of Circles

Theorem 14 (Circle Equations-Standard Form)

An equation for a circle with center (h, k) and radius r is equivalent to

$$(x - h)^2 + (y - k)^2 = r^2.$$

Equations of Circles

Example 15

Find an equation for the circle with center $(2, -1)$ that passes through the point $(4, -6)$.

Equations of Circles

Example 16 (Do you understand circle equations?)

What is the center of the circle whose equation in standard form is $(x - 3)^2 + y^2 = 25$?

- (a) $(3, 5)$
- (b) $(0, 3)$
- (c) $(3, 0)$
- (d) $(-3, 0)$
- (e) The graph of this equation is not a circle.

Sketch the circle defined by $(x - 3)^2 + y^2 = 25$.

Equations of Circles

Example 17

Is the graph of $x^2 + 10x + y^2 - 6y + 32 = 0$ a circle? If so, find its center and radius.

Equations of Circles

Example 18

Is the graph of $x^2 + 4x + y^2 - 2y + 40 = 0$ a circle? If so, find its center and radius.

Midpoints

Definition 19 (Midpoint)

The **midpoint** of the line segment AB is the point on the line segment that is equidistant from A and B .

Midpoints

Example 20

Suppose that the distance from A to B equals the distance from B to C . Is it necessarily true that B is the midpoint of AC ? Why or why not?

What extra piece of information do you need?

Midpoints

Example 21

Consider the points $A(2, 0)$, $B(0, 5)$, and $C(-2, 0)$. Is B the midpoint of AC ?

Midpoints

Theorem 22

The midpoint of the line segment from (x_1, y_1) to (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Proof.

This proof is left as an exercise. □

Midpoints

Example 23 (Circles and Diameters)

A diameter of a circle has endpoints $(2, 5)$ and $(4, -1)$. Find an equation for the circle.

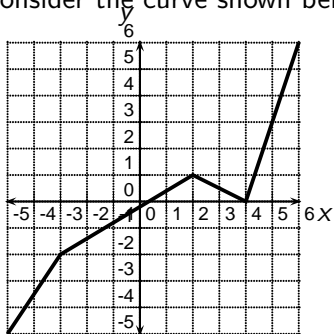
Steepness, Lines, and Rates of Change

The distance formula allows us to measure the length of a line segment in two dimensions. Distance or length is just one of the useful quantities we can measure in two dimensions. The steepness of a line segment is another useful measurement in two dimensions. How fast is a line segment rising or falling?

Steepness, Lines, and Rates of Change

Example 24

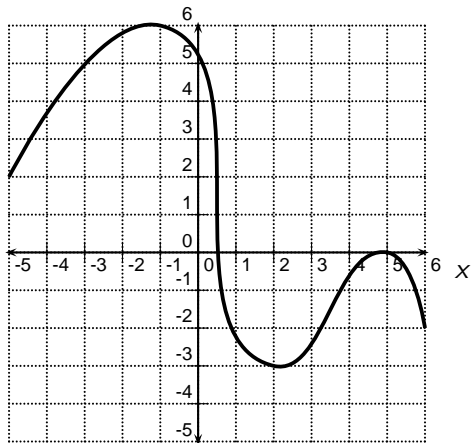
Consider the curve shown below. Discuss the steepness of the curve.



Steepness, Lines, and Rates of Change

Example 25

Consider the curve shown below. Discuss the steepness of the curve.



Steepness, Lines, and Rates of Change

Example 26

Suppose you want to draw a curve for which the steepness does not vary. What type of curve could you draw?

We have been discussing steepness in very vague qualitative terms. If we want to be able to make more concrete comparisons, we need to be able to quantify, or measure, the steepness of a curve. You will need Calculus in order to measure the steepness of general curves like the curve shown in Example 25. Today we will quantify the steepness of lines because they are the simplest curves. Moreover, this measurement will lay a foundation for what you will do in Calculus.

Quantifying the Steepness of a Line - Slope

The steepness of a line is determined by comparing the change in the vertical distance to the change in the horizontal distance. The ratio of the change in vertical distance to the change in horizontal distance is constant for a line. For a line, this ratio is called the **slope** of the line.

Definition 27

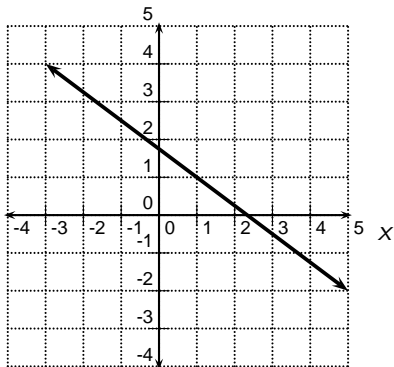
If $x_1 \neq x_2$, then the slope of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Quantifying the Steepness of a Line - Slope

Example 28

Find the slope of the line shown below.



Quantifying the Steepness of a Line - Slope

Example 29

Find the slope of the line that passes through $(6, 2)$ and $(6, 5)$.

Quantifying the Steepness of a Line - Slope

Example 30

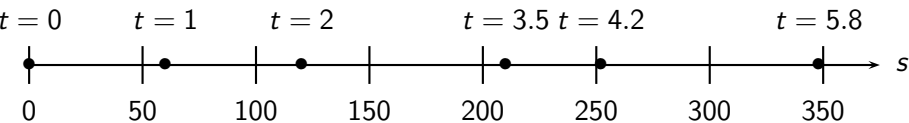
Find the slope of the line that passes through $(1, 3)$ and $(-5, 3)$.

Rates of Change

The slope of a line is a **rate of change**. If the line is in terms of the variables x and y , then the slope of the line is the rate of change of y with respect to x . This means that the slope is the ratio of the change in y to the change in x . The word “**per**” is often associated with rates of change. Think about speed. Speed can be measured in miles **per** hour. Speed is a rate of change of distance with respect to time.

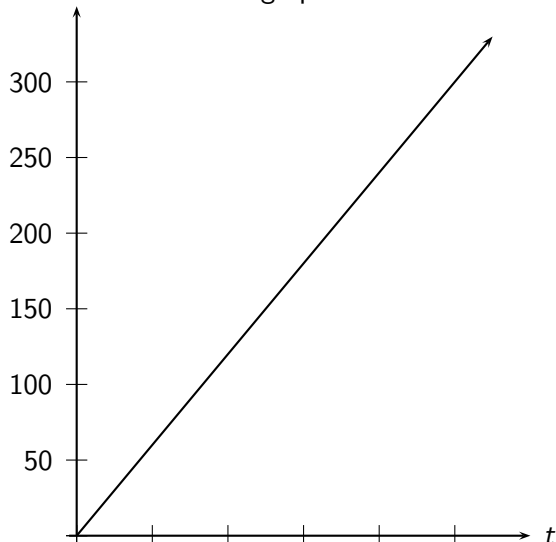
Rates of Change

A particle is traveling along a straight line. Its position, s , at time t seconds is given by $s = 60t$ where s is measured in feet.



Rates of Change

The two-dimensional graph of $s = 60t$ is shown below. This graph is linear.



Rates of Change

Example 31 (Rate of Change)

1. What is the slope of this line?
2. What are the units of the slope?
3. Express the slope of the line as a rate of change.
4. What does the slope of this line tell us about the particle?

Equations of Lines

As we said before, the slope of a non-vertical line is constant. A line can be totally described by its slope and a point on the line. Why do we need a point? Why is the slope not sufficient?

Equations of Lines

Example 32

Find an equation for the line that passes through $(4, 3)$ and has slope -2 .

Equations of Lines

Example 33

Find an equation for the line that passes through (a, b) and has slope -2 .

Equations of Lines

Theorem 34 (Equations of Lines-Point Slope Formula)

The graph of

$$y - b = m(x - a)$$

is a line that passes through (a, b) and has slope m .

Equations of Lines

Theorem 35 (Equations of Lines-Horizontal and Vertical Lines)

The graph of $y = b$ is a horizontal line that passes through (a, b) . The slope of a horizontal line is 0. The graph of $x = a$ is a vertical line that passes through (a, b) . The slope of a vertical line is undefined.

Equations of Lines

Definition 36 (Linear Equations)

A **linear equation** in x and y is an equation that is equivalent to an equation of the form

$$Ax + By + C = 0$$

where A , B , and C are constants.

Equations of Lines

Example 37

Verify that $y - b = m(x - a)$ is a linear equation. What are the values of A , B , and C in the definition of a linear equation.

Equations of Lines

Example 38 (Concept Check)

Which of the following are linear equations?

A. $2x + 4y = 5$

B. $y - 3 = 4(x - 2)$

C. $y = \frac{2}{x + 5}$

D. $x = 4y - 3$

E. $y = -\frac{2}{3}x$

F. $(x - 1)^2 + (y + 2)^2 = 3$

Miscellaneous Information About Lines

The following facts about lines are **IMPORTANT**:

- Slope-intercept form ($y = mx + b$) is **NOT** the only way to write an equation for a line. Sometimes it is not even the most useful way to write an equation for a line.
- Point-slope form is often more useful than slope-intercept form. **Learn it!**
- Slope-intercept form is useful if you need to determine the slope of a line or if you are checking to see if two lines are the same.

Miscellaneous Information About Lines

The following facts about lines are **IMPORTANT**:

- Parallel lines have the same slope.
- The product of the slopes of perpendicular lines is -1 .
- The slope of vertical lines ($x = a$) is undefined.
- The slope of horizontal lines ($y = a$) is zero.
- To find the equation of a line, you need the slope and a point.

Miscellaneous Information About Lines

Example 39

Find an equation for the line that passes through $(-9, 5)$ and is parallel to the line whose equation is $y - 2 = \frac{7}{4}(x + 1)$.

Miscellaneous Information About Lines

Example 40

Find an equation for the line that is perpendicular to $2x + 3y - 5 = 0$ and passes through the point $(2, -1)$.

Miscellaneous Information About Lines

Example 41

Find the slope of the line given by the equation $4x + 5y = 10$? What is its y -intercept? Graph the line.