5 Systems of Equations

Concepts:

- Solutions to Systems of Equations-Graphically and Algebraically
- Solving Systems - Substitution Method
- Solving Systems - Elimination Method
- Using 2-Dimensional Graphs to Approximate Solutions of Equations in One Variable.
  - The Intersection Method
  - The Intercept Method

(Sections 11.1, 11.1A)

Thus far we have been looking at single equations. Often times we have more than just one constraint on our variables which produces more than one equation.

5.1 Solutions to Systems of Equations

A **system of equations** is a set of equations involving the same variables. For example,

$$\begin{aligned} 2x + y &= 5 \\ x - y &= 4 \end{aligned} \qquad (1)$$

is a system of equations in the variables x and y.

A **solution of a system** is a solution of ALL the equations in the system at the same time. Often times when a system involves the variables x and y, we write the solution as a point.

Example 5.1
Verify that $(3, -1)$ is a solution to the system in (1).

$2(3) + (-1) \overset{?}{=} 5$

$6 + (-1) \overset{?}{=} 5$

$5 = 5$ ✓

?

$3 - (-1) \overset{?}{=} 4$

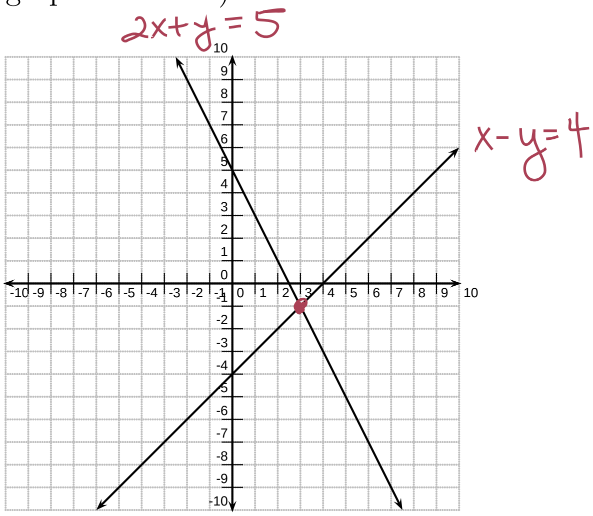
$3 + 1 \overset{?}{=} 4$

$4 = 4$ ✓

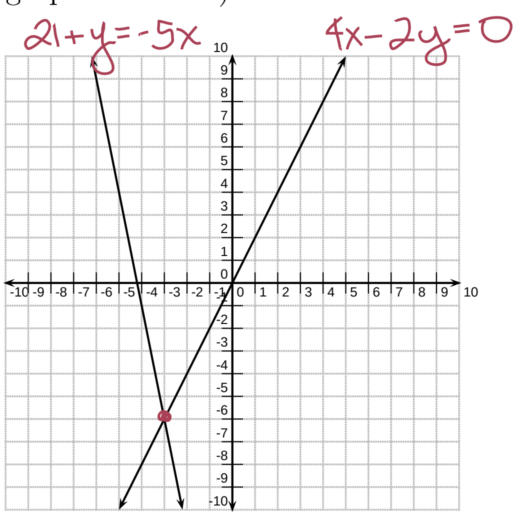
$(3,-1)$ is a solution to both eqtns $\Rightarrow (3,-1)$ is a solution to the system.

Recall that we visualize the set of solutions to a single equation in two variables as a graph. We visualize a solution to a system of equations as the set of points that are on the graphs of all the equations in the system at the same time. In other words, we visualize a solution to a system of equations as an intersection point.

Example 5.2
Below are the graphs of the equations $2x + y = 5$ and $x - y = 4$. Notice the two graphs intersect at the point $(3, -1)$, which is also the solution to the system. (Can you tell which graph is which?)



Example 5.3 (Do you understand solutions to systems?)
Below are the graphs of the equations $4x - 2y = 0$ and $21 + y = -5x$. (Can you tell which graph is which?)



How many solutions does the system of equations below have? Use the graph to find approximate solution(s). Verify your solution(s) algebraically.

$$\begin{array}{rcrcr} 4x & - & 2y & = & 0 \\ 21 & + & y & = & -5x. \end{array}$$

1 solution
$\approx (-3, -6)$

Check algebraically,

$4(-3) - 2(-6) \stackrel{?}{=} 0$

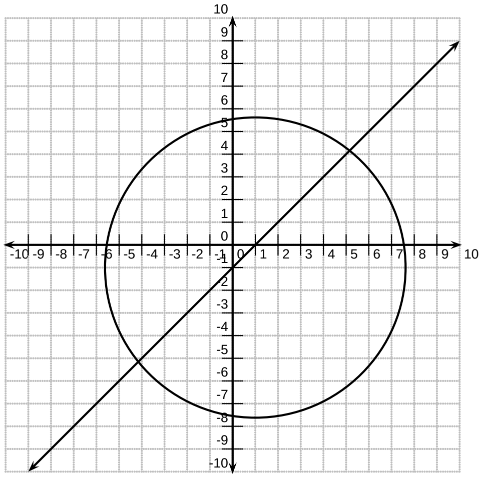
$-12 + 12 = 0$ ✓

$21 + (-6) \stackrel{?}{=} -5(-3)$

$15 = 15$ ✓

$(-3,-6)$ is the solution to the system.

Example 5.4 (Do you understand solutions to systems?)
Below are the graphs of the equations $(x-1)^2+(y+1)^2 = 41$ and $x - y = 1$.



2 solutions

$\approx (5,4)$ and $(-4,-5)$

How many solutions does the system of equations below have? Use the graph to find approximate solution(s). Verify your solution(s) algebraically.

$$\begin{aligned}(x-1)^2+(y+1)^2 &= 41\\ x-y &= 1.\end{aligned}$$

Algebraically check.

$(5-1)^2+(4+1)^2 \stackrel{?}{=} 41$

$16 + 25 = 41$ ✓

$5-4=1$ ✓

$(-4-1)^2+(-5+1)^2 \stackrel{?}{=} 41$

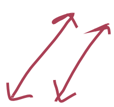
$25+16 = 41$ ✓

$-4-(-5) \stackrel{?}{=} 1$

$-4+5 = 1$ ✓

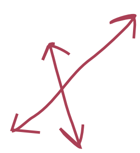
Example 5.5 (Do you understand solutions to systems?)
Suppose you have a system of equations where the graphs of both equations are lines. How many solutions could you possibly have?

0

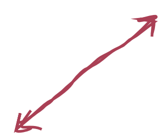


parallel lines

1



infinitely many



2 lines coincide

(Note: Systems of equations in which every equation in the system is linear are called *linear systems*.)

It is useful to know how to visualize solutions to a system. However, the only way to guarantee an **exact** solution is to solve the equation **algebraically**. Solutions that are obtained by **graphical** means are **approximations**.

5.2 Solving Systems of Equations - Substitution Method

There are two methods of solving systems that we will study. We begin with the substitution method. Let us illustrate this with an example.

Example 5.6 (Substitution Method)
Given the system of equations,

$$\begin{array}{rcrcr} 4x & - & 2y & = & 0 \\ 21 & + & y & = & -5x \end{array} \longrightarrow y = -5x - 21$$

we know that any solution to this system must satisfy the second equation. Solve this system of equations by solving the second equation for y and substituting the result in for y in the first equation.

$4x - 2(-5x-21) = 0$

$4x + 10x + 42 = 0$

$14x + 42 = 0$

$14x = -42$

$x = -3$

$y = -5(-3) - 21 = -6$

$(-3, -6)$ is the only solution to the system.

How many solutions does the system have?

The Substitution Method

1. Solve one equation for one the variables.
2. Substitute it into the OTHER equation.
3. Once you have a value for one of your variables, substitute it into one of your original equations to solve for the other variable.

Example 5.7 (Substitution Method)
Use the substitution method to solve the system of equations below. Express your solution as a point.

$$\begin{array}{rcrcl} 2x & - & y & = & 1 \\ 3x & + & 2y & = & 4 \end{array} \qquad y = 2x - 1$$

$3x + 2(2x-1) = 4$

$3x + 4x - 2 = 4$

$7x = 6$

$x = \frac{6}{7}$

$y = 2\left(\frac{6}{7}\right) - 1 = \frac{12}{7} - 1 = \frac{5}{7}$

$\left(\frac{6}{7}, \frac{5}{7}\right)$ is the only solution to the system.

Example 5.8 (Substitution Method)
Use the substitution method to solve the system of equations below.

$$\begin{array}{rcl} (x-1)^2 + (y+1)^2 & = & 41 \\ x - y & = & 1. \end{array} \qquad x = y + 1$$

$(y+1-1)^2 + (y+1)^2 = 41$

$y^2 + y^2 + 2y + 1 = 41$

$2y^2 + 2y - 40 = 0$

$y = \dfrac{-2 \pm \sqrt{4 - 4(2)(-40)}}{2(2)} = \dfrac{-2 \pm \sqrt{324}}{4} = \dfrac{-2 \pm 18}{4} = 4, -5$

$y = 4$
$x = 4 + 1 = 5$

$y = -5$
$x = -5 + 1 = -4$

$(5, 4)$ and $(-4, -5)$ are the solutions to the system.

How many solutions does the system have?

2 solutions

Example 5.9 (Substitution Method)
Use the substitution method to solve the system of equations below.

$$\begin{array}{rcrcr} 3x & - & 6y & = & 9 \\ 2x & - & 4y & = & 6 \end{array} \rightarrow x = 2y+3$$

$3(2y+3) - 6y = 9$

$6y + 9 - 6y = 9$

$9 = 9$

Always true!

Can find a solution for any y.

How many solutions does the system have? What are three solutions to the system?

infinitely many solutions $(3,0)$, $(0,-\frac{3}{2})$, $(1,-1)$ are all solutions.

When a system of equations involves two equations that are equivalent to each other, we call that system **dependent**.

5.3 Solving Systems of Equations - Elimination Method

The second method for solving systems of equations is the elimination method. Recall that when you add or multiply the same quantity to both sides of an equation, the result is an equivalent equation. Again, we begin with an example.

Example 5.10 (Elimination Method)
Given the system of equations,

$$\begin{array}{rcrcr} 5x & - & 3y & = & 0 \\ 4x & - & 6y & = & -6 \end{array}$$

solve the system by adding a multiple of the first equation to the second equation in order to eliminate the y variable.

Multiply top eqtn by -2.

$$\begin{array}{rcl} -10x + 6y & = & 0 \\ 4x - 6y & = & -6 \\ \hline -6x & = & -6 \end{array}$$

$x = 1$

$4(1) - 6y = 0$

$4 - 6y = 0$

$-6y = -4$

$y = \frac{2}{3}$

$(1, 2/3)$ is the only solution.

The Elimination Method

1. Adjust the coefficients of each equation so that the coefficients of one of the variables are additive inverses.
2. Add the two equations together to eliminate one of the variables.
3. Solve for the non-eliminated variable.
4. Once you have a value for one of your variables, substitute it into one of your original equations to solve for the other variable.

Example 5.11 (Elimination Method)
Use the elimination method to solve the system of equations below.

$$\begin{array}{rcrcl} 2x & - & y & = & 1 \\ 3x & + & 2y & = & 4 \end{array}$$

$$\begin{array}{r} 4x-2y=2 \\ 3x+2y=4 \\ \hline 7x=6 \\ x=6/7 \end{array}$$

$$2\left(\frac{6}{7}\right)-y=1$$
$$-y=\frac{-5}{7}$$
$$y=5/7$$

$\left(\frac{6}{7},\frac{5}{7}\right)$ is the solution

Example 5.12 (Elimination Method)
Use the elimination method to solve the system of equations below.

$$\begin{array}{rcrcl} x^3 & + & 4y^2 & = & 12 \\ x & + & y^2 & = & 3. \end{array}$$

$$\begin{array}{r} x^3+4y^2=12 \\ -4x-4y^2=-12 \\ \hline x^3-4x=0 \end{array}$$

$$x(x^2-4)=0$$

$x=0$ or $x^2-4=0$

$x^2=4 \rightarrow x=\pm 2$

$$\begin{array}{c} x=0 \\ \hline y^2=3 \\ y=\pm\sqrt{3} \end{array}$$

$$\begin{array}{c} x=2 \\ \hline 2+y^2=3 \\ y^2=1 \\ y=\pm 1 \end{array}$$

$$\begin{array}{c} x=-2 \\ \hline -2+y^2=3 \\ y^2=5 \\ y=\pm\sqrt{5} \end{array}$$

How many solutions does the system have?

6 solutions

$(0,\sqrt{3})$ $(2,1)$ $(-2,\sqrt{5})$

$(0,-\sqrt{3})$ $(2,-1)$ $(-2,-\sqrt{5})$

Example 5.13 (Elimination Method)
Use the elimination method to solve the system of equations below.

$$\begin{array}{rcrcr} 12x & - & 6y & = & 0 \\ 10x & - & 5y & = & 15 \end{array}$$

← multiply by 5
← mult. by -6

$60x-30y=0$
$-60x+30y=-90$

$0=-90$ ← Never true no matter what x&y are.

How many solutions does the system have? 0 solutions.

Systems of equations in which there are no solutions are called **inconsistent**. An example of an inconsistent system is one where the graphs of the equations are parallel lines. Parallel lines never intersect and thus, there is no solution to the system of equations.

Example 5.14 (Points of Intersection)
Find the points of intersection between the graphs of $2x + y - 4 = 0$ and $(x + 1)^2 + y^2 = 9$. Which method should you use to solve this system? Substitution

$2x+y-4=0$
$y=4-2x$

$(x+1)^2+(4-2x)^2=9$
$x^2+2x+1+16-16x+4x^2=9$
$5x^2-14x+8=0$
$(5x-4)(x-2)=0$
$5x-4=0$ $\quad$ $x-2=0$
$5x=4$ $\quad$ $x=2$
$x=\frac{4}{5}$

$x=\frac{4}{5}$
$y=4-2\left(\frac{4}{5}\right)=\frac{12}{5}$
$\left(\frac{4}{5}, \frac{12}{5}\right)$

$x=2$
$y=4-2(2)=0$
$(2,0)$

$\boxed{\text{2 solutions } \left(\frac{4}{5}, \frac{12}{5}\right) \text{ and } (2,0)}$

5.4 Using 2-Dimensional Graphs to Approximate Solutions of Equations in One Variable

When you graph an equation on the Cartesian Coordinate System, you are displaying all of the (x, y) solutions of that equation. At first glance, it may not seem that the Cartesian Coordinate System would be very helpful for envisioning the solutions of an equation in one variable such as $x^3 = 3x^2 - 7$. If we introduce the variable y in an appropriate way, we will be able to approximate the solutions of this equation by looking at a graph on the Cartesian Coordinate System. The next example will illustrate two different methods for approximating solutions graphically.

> Remember, the only way to guarantee an **exact** solution is to solve the equation **algebraically**. Solutions that are obtained by **graphical** means are **approximations**.

Example 5.15 (The Intersection and Intercept Methods)

(a) Solve the equation.

$$3x^2 = 2 - 5x$$

$3x^2 + 5x - 2 = 0$

$(3x - 1)(x + 2) = 0$

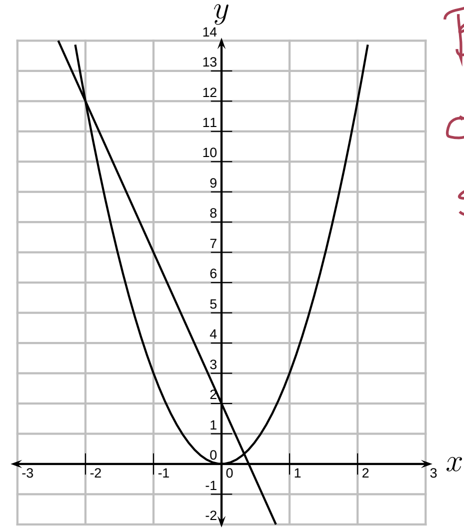
$3x - 1 = 0$ or $x + 2 = 0$

$3x = 1$ $\quad$ $x = -2$

$x = \frac{1}{3}$

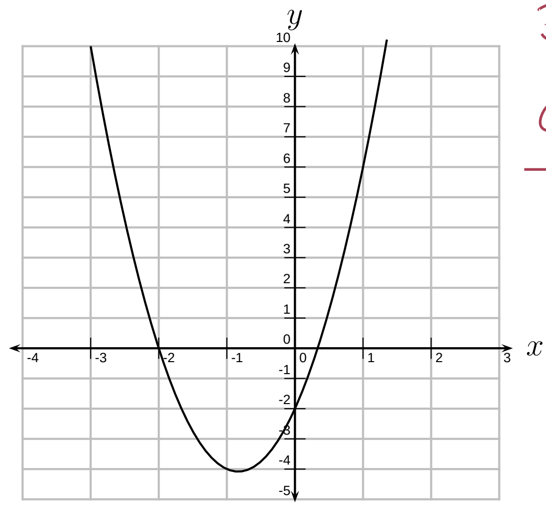
$\frac{1}{3}, 2$

(b) (*The Intersection Method*) The graphs of $y = 3x^2$ and $y = 2 - 5x$ are shown below. Explain how these graphs can be used to approximate the solutions of $3x^2 = 2 - 5x$.



By using substitution, the x-coor. of the solutions to the system $\begin{cases} y = 3x^2 \\ y = 2 - 5x \end{cases}$ are exactly the solutions to $3x^2 = 2 - 5x$. Thus, the x-coor. of the intersection points of the graphs are the solutions.

(c) (*The Intercept Method*) The graph of $y = 3x^2 + 5x - 2$ is shown below. Explain how this graphs can be used to approximate the solutions of $3x^2 = 2 - 5x$.



$3x^2 = 2 - 5x$ is equivalent to $0 = 3x^2 + 5x - 2$.
The x-intercepts of $y = 3x^2 + 5x - 2$ are exactly the solutions to $0 = 3x^2 + 5x - 2$ and thus solutions to $3x^2 = 2 - 5x$.

(d) Explain how the Intercept Method is related to the Intersection Method.

The Intercept Method is a special case of the Intersect Method, just letting one of the equations be $y = 0$.