

11 Polynomials

Concepts:

- Quadratic Functions
 - The Definition of a Quadratic Function
 - Graphs of Quadratic Functions - Parabolas
 - Vertex
 - Absolute Maximum or Absolute Minimum
 - Transforming the Graph of $f(x) = x^2$ to Obtain the Graph of Any Quadratic Function.

- Polynomial Functions
 - The Definition of a Polynomial
 - Leading Coefficient, Leading Term, Constant Term and the Degree of a Polynomial
 - Graphs of Polynomials
 - * Basic Shapes
 - * Continuous Graphs
 - * Smooth Graphs
 - * End Behavior of the Graph
 - * Multiplicity of a Root and Behavior of the Graph at x -intercepts.
 - * How Many Local Extrema Can a Polynomial Graph Have?
 - Polynomial Division - The Division Algorithm
 - Roots and Zeros of a Polynomial
 - The Remainder Theorem
 - The Factor Theorem
 - The Rational Root Theorem
 - Finding Rational Roots of a Polynomial
 - Finding Real Roots of a Polynomial

(Sections 4.1-4.4)

We have already studied linear functions and power functions. Linear and power functions are special types of functions known as polynomial functions. Polynomial functions have several very nice properties that we will look at at the end of this section. One property to notice though, is that the domain of a polynomial function is $(-\infty, \infty)$.

Polynomial functions can be simply built from the power functions we have already studied. Linear functions are the simplest of all polynomial functions. Quadratic functions are next in the line of polynomial functions. Although quadratic functions are a bit more complicated than linear functions, we gain a lot by moving up a level as you will see when we discuss optimization.

11.1 Quadratic Functions

Definition 11.1

A **quadratic function** is a function that is equivalent to a function of the form

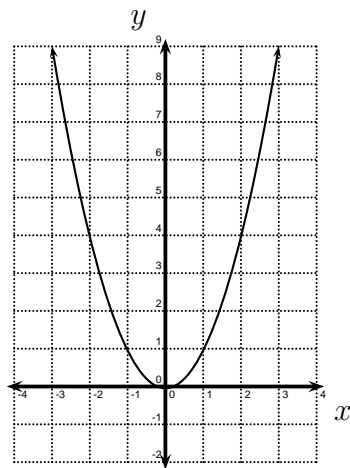
$$q(x) = ax^2 + bx + c$$

where a , b , and c are constants and $a \neq 0$.

Definition 11.2

The graph of a quadratic function is called a **parabola**.

The most basic quadratic function is $f(x) = x^2$. The graph of f is shown below.



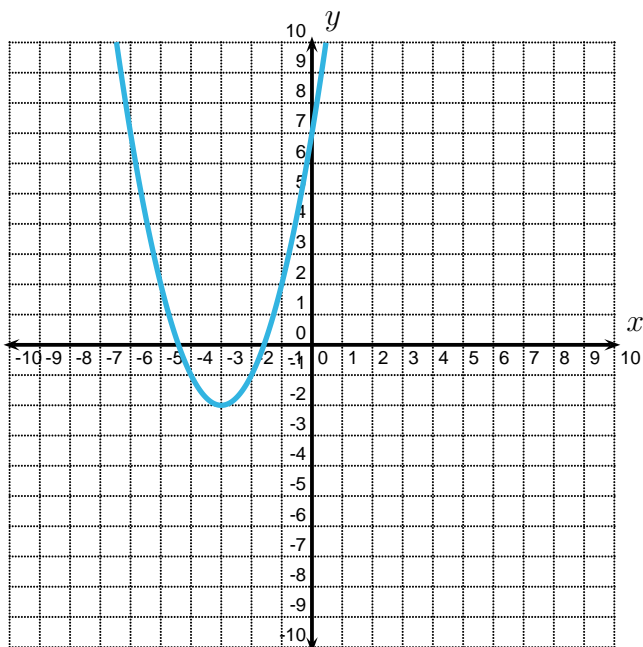
Parabolas are important because they have either an **absolute minimum value** (a smallest output value) or an **absolute maximum value** (a largest output value). The point on the graph that corresponds to the absolute minimum or absolute maximum value is called the **vertex** of the parabola. For the graph above, the absolute minimum value is 0 and the vertex is $(0, 0)$.

The graph of every quadratic function can be obtained by transforming the graph of $y = f(x) = x^2$.

Example 11.3 (Transformations and Quadratic Functions)

Let $f(x) = x^2$ and $g(x) = x^2 + 6x + 7$.

- Describe the transformations that could be applied to the graph of f to obtain the graph of g . **Shift left 3 units and down 2 units.**
- Sketch the graph of g .
- What is the vertex of the graph of g ? **The vertex is $(-3, -2)$**
- Does g have an absolute minimum value or an absolute maximum value? What is it? **The function g has an absolute minimum value of -2 at $x = -3$.**



Note,

$$\begin{aligned}g(x) &= x^2 + 6x + 7 \\&= (x^2 + 6x + 9) - 9 + 7 \\&= (x + 3)^2 - 2 \\&= f(x + 3) - 2\end{aligned}$$

So, the the graph of g is the graph of f shifted left 3 units and down 2 units.

Definition 11.4

A quadratic function $q(x) = ax^2 + bx + c$ can be rewritten in its **standard form**

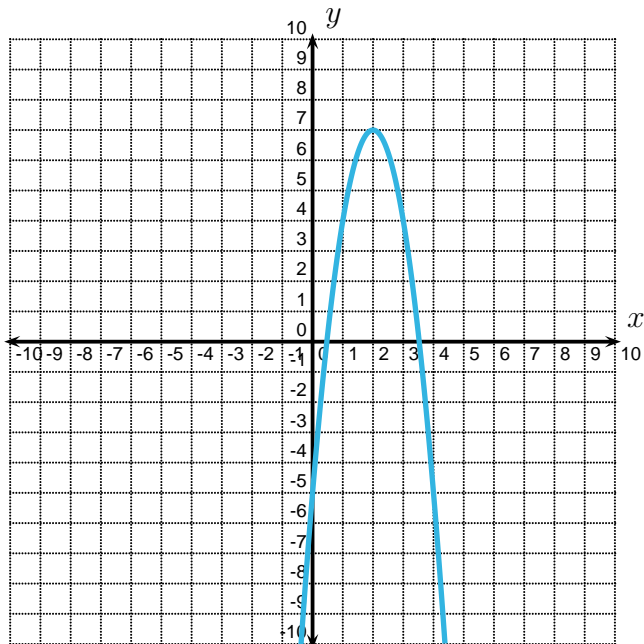
$$q(x) = a(x - h)^2 + k$$

where $a \neq 0$ and (h, k) is the vertex of the parabola.

Example 11.5 (Transformations and Quadratic Functions)

Let $f(x) = x^2$ and $g(x) = -3x^2 + 12x - 5$.

- Express $g(x)$ in standard form and sketch its graph.
- What is the vertex of the graph of g ? (2, 7)
- Does g have an absolute minimum value or an absolute maximum value? What is it?
 g has an absolute maximum value of 7 at $x = 2$.



$$\begin{aligned}g(x) &= -3x^2 + 12x - 5 \\&= (-3x^2 + 12x) - 5 \\&= -3(x^2 - 4x) - 5 \\&= -3(x^2 - 4x + 4) - 4(-3) - 5 \\&= -3(x - 2)^2 + 7\end{aligned}$$

Example 11.6 (Min or Max?)

Given a quadratic function $f(x) = ax^2 + bx + c$,

- when does a quadratic function have an absolute maximum value?

when $a < 0$

- when does a quadratic function have an absolute minimum value?

when $a > 0$

- The vertex of a quadratic function occurs when $x = -\frac{b}{2a}$.
What is the y -coordinate of the vertex?

$$f\left(\frac{-b}{2a}\right)$$

Example 11.7

Find the maximum value of the function $f(x) = -3x^2 + 10x + 4$.

The since $a < 0$, then $f(x)$ has an absolute maximum at the vertex. The maximum value of f is the y -coordinate of the vertex $x = \frac{-b}{2a} = \frac{-10}{-6} = \frac{5}{3}$ which is

$$f\left(\frac{5}{3}\right) = -3\left(\frac{5}{3}\right)^2 + 10\left(\frac{5}{3}\right) + 4 = \frac{37}{3}$$

Example 11.8

Find a quadratic function $f(x) = ax^2 + bx + c$ whose vertex is $(3, -1)$ and goes through the point $(5, 7)$. Using the standard form of $f(x) = a(x - h)^2 + k$ with vertex $(3, -1)$ gives

$$f(x) = a(x - 3)^2 - 1$$

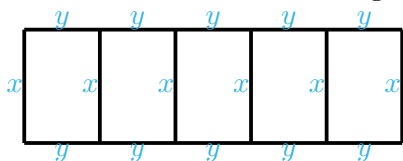
Since $(5, 7)$ is a point on the parabola, then $f(5) = 7$, or

$$\begin{aligned} a(5 - 3)^2 - 1 &= 7 \\ a(2)^2 &= 8 \\ a &= 2 \end{aligned}$$

Hence, $f(x) = 2(x - 3)^2 - 1 = 2(x^2 - 6x + 9) - 1 = 2x^2 - 12x + 17$

Example 11.9 (Optimization)

A farmer has 200 feet of fencing to construct five rectangular pens as shown in the diagram below.



What is the maximum possible area of all five pens?

Let x be the length of each pen and y be the width of each pen as indicated in the diagram above. The area, A , of each pen is given by

$$A = xy$$

and, the amount of fencing it takes to create the pens is given by

$$6x + 10y$$

Since we want to maximize the total area, then we should use all of the fencing we have available. That is,

$$6x + 10y = 200$$

Solving this constraint for y gives

$$y = 20 - \frac{6}{10}x$$

So, the area of each pen can be written as

$$A = 5xy = 5x\left(20 - \frac{6}{10}x\right) = 100x - 5\frac{6}{10}x^2 = -3x^2 + 100x + 0$$

This is a quadratic function that has an absolute maximum at $x = \frac{-b}{2a} = \frac{-100}{2(-3)} = \frac{50}{3}$ and hence

$$\text{Abs Max} = -3 \left(\frac{50}{3} \right)^2 + 100 \frac{50}{3} = -\frac{2500}{3} + \frac{5000}{3} = \frac{2500}{3} \approx 833.33\text{ft}^2$$

11.2 Polynomials

Definition 11.10

A **polynomial** in x is an algebraic expression that is equivalent to an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a non-negative integer, x is a variable, and the a_i 's are all constants.

In a polynomial, a_i is called the **coefficient** of x^i and a_0 is called the **constant term** of the polynomial. If the polynomial contains only a constant term, it is called a **constant polynomial**. If the polynomial equals 0, then it is called the **zero polynomial**.

If you write the non-zero terms in order so the exponents of x decrease, the first term $a_n x^n$ is the **leading term**. The coefficient of the leading term, a_n , is called the **leading coefficient**. The exponent of x in the leading term is called the **degree** of the polynomial. The degree of the zero polynomial is undefined since there is no nonzero term.

Polynomial	Not A Polynomial
$x^3 + 3x^4 + 2$	$x^{-3} + 3x^4 + 2$
$\sqrt{2}x + 1$	$\sqrt{2x} + 1$

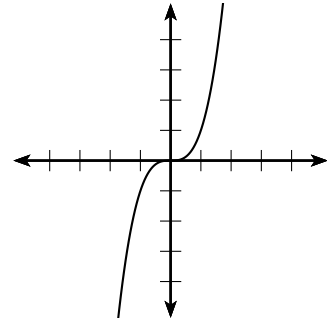
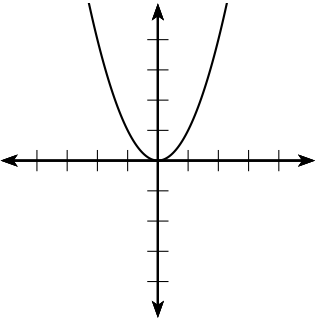
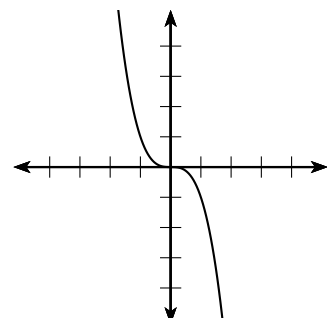
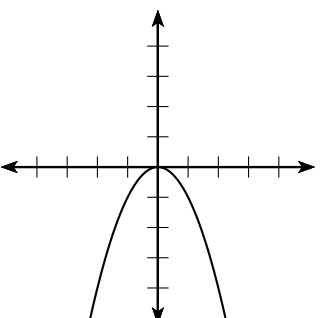
Example 11.11 (Polynomials)

Determine which of the following expressions are polynomials. If the expression is a polynomial, find its leading term, leading coefficient and degree.

	Polynomial?	Leading Term	Leading Coefficient	Degree
$\frac{1}{5}x^7 + 2x^3 + \frac{2}{9}x + x^8 + 3x^4$	Yes	x^8	1	8
$x^{\frac{1}{3}} + 3x^4 + 2$	No			
$(1-x)^4(5+3x)^2$	Yes	$(-x)^4(3x)^2 = x^4(9x^2) = 9x^6$	9	6

11.2.1 Graphs of Polynomials

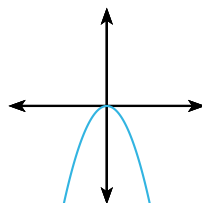
Basic Shapes Let $f(x) = ax^n$. Below is a chart of the basic shapes that the graph of $f(x)$ can take on.

	n odd	n even
a positive		
a negative		

Example 11.12

What is the basic shape of $f(x) = -3x^4$?

Note that $a = -3$ is negative and $n = 4$ is even.



Special Properties of Polynomial Graphs

(P-1) Polynomial Graphs are **continuous**.

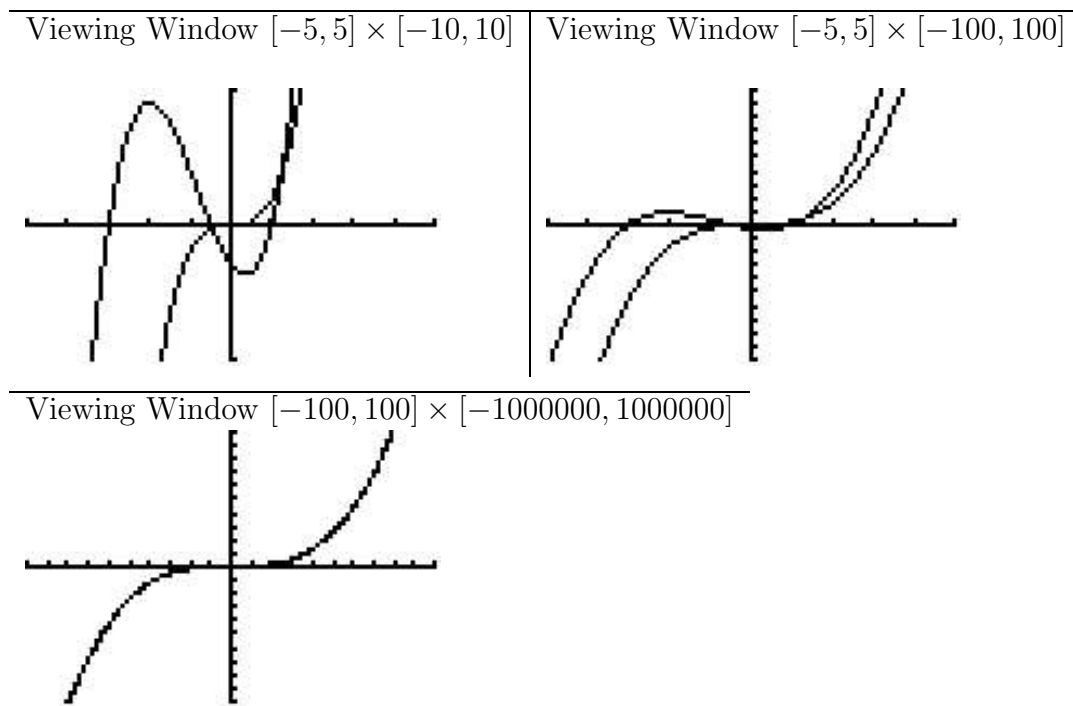
When you study Calculus, you will study the precise definition of a continuous function. At this point in your studies, you can begin to understand the idea that lead to the precise definition. The graph of a functions is **continuous** if there are no breaks in the graph. Intuitively, this means that you can sketch the graph without picking up your pencil. (The graph of the greatest integer function is not continuous because there are breaks in the graph.)

(P-2) Polynomial graphs are **smooth**.

The graph of a polynomial does not have any sharp corners. (The graph of the absolute value function is not smooth because there is a sharp corner at the tip of the vee.)

(P-3) The shape of a polynomial graph is dominated by the leading term when $|x|$ is large.

For example, let's consider the graphs of $y = 2x^3$ and $y = 2x^3 + 5x^2 - 4x - 3$.



We say that the graphs of $y = 2x^3$ and $y = 2x^3 + 5x^2 - 4x - 3$ have the same **end behavior**. Eventually, the leading term of any polynomial will dominate the shape of its graph, so the end behavior of a polynomial graph can be determined by examining the graph of the leading term.

END BEHAVIOR:

We have some notation to help us describe the end behavior of graphs.

- If the y values become very large as x becomes very large, we denote this by

$$y \rightarrow \infty \text{ as } x \rightarrow \infty.$$

- If the y values become very large and negative as x becomes very large, we denote this by

$$y \rightarrow -\infty \text{ as } x \rightarrow \infty.$$

- If the y values become very large as x becomes very large and negative, we denote this by

$$y \rightarrow \infty \text{ as } x \rightarrow -\infty.$$

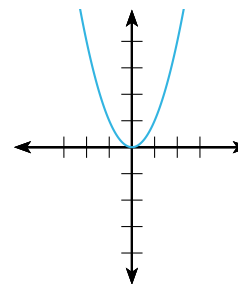
- If the y values become very large and negative as x becomes very large and negative, we denote this by

$$y \rightarrow -\infty \text{ as } x \rightarrow -\infty.$$

Example 11.13 (End Behavior)

Describe the end behavior of the graph of $y = 7 - x + 3x^8$.

The end behavior of $y = 7 - x + 3x^8$ is dominated by the leading term $3x^8$. Since $a = 3$ is positive and $n = 8$ is even, then the graph of y has the basic shape shown to the right.



As you can see, $y \rightarrow \infty$ as $x \rightarrow \infty$ and $y \rightarrow \infty$ as $x \rightarrow -\infty$.

Example 11.14 (End Behavior)

Describe the end behavior of the graph of $y = ax^n$ by examining the basic shapes.

- If $a > 0$ and n even,

$$y \rightarrow \underline{\infty} \text{ as } x \rightarrow \infty \text{ and}$$

$$y \rightarrow \underline{\infty} \text{ as } x \rightarrow -\infty.$$

- If $a < 0$ and n even,

$$y \rightarrow \underline{-\infty} \text{ as } x \rightarrow \infty \text{ and}$$

$$y \rightarrow \underline{-\infty} \text{ as } x \rightarrow -\infty.$$

- If $a > 0$ and n odd,

$$y \rightarrow \underline{\infty} \text{ as } x \rightarrow \infty \text{ and}$$

$$y \rightarrow \underline{-\infty} \text{ as } x \rightarrow -\infty.$$

- If $a < 0$ and n odd,

$$y \rightarrow \underline{-\infty} \text{ as } x \rightarrow \infty \text{ and}$$

$$y \rightarrow \underline{\infty} \text{ as } x \rightarrow -\infty.$$

Example 11.15

Describe the end behavior of the graph of $y = (5 - x)^3(2x + 15)^{71}$.

The leading term of $y = (5 - x)^3(2x + 15)^{71}$ is $(-x)^3(2x)^{71} = -x^3(2)^{71}x^{71} = -2^{71}x^{74}$. So $a = -2^{71}$ is negative and $n = 74$ is even. Thus the end behavior of the graph is

$$y \rightarrow -\infty \quad \text{as} \quad x \rightarrow \infty$$

$$y \rightarrow -\infty \quad \text{as} \quad x \rightarrow -\infty$$

(P-4) The graph of a polynomial of degree n can have **at most** n x -intercepts.

Definition 11.16 (Roots and Zeros)

Let $P(x)$ be a polynomial. The number c is called a **root** or a **zero** of P if and only if $P(c) = 0$. (Note that a zero is the same as an x -intercept.)

Example 11.17 (To Cross or To Touch and Turn Around?)

Use your graphing calculator to look at the following graphs and describe the behavior of the graph at $x = 3$ and $x = -5$.

- (a) $y = (x - 3)(x + 5)$ The graph crosses the x axis at $(3,0)$ and at $(-5,0)$
- (b) $y = (x - 3)^2(x + 5)$ The graph touches the x axis at $(3,0)$ and crosses at $(-5,0)$
- (c) $y = (x - 3)^3(x + 5)$ The graph crosses the x axis at $(3,0)$ and at $(-5,0)$
- (d) $y = (x - 3)^4(x + 5)$ The graph touches the x axis at $(3,0)$ and crosses at $(-5,0)$
- (e) $y = (x - 3)(x + 5)^2$ The graph crosses the x axis at $(3,0)$ and touches at $(-5,0)$
- (f) $y = (x - 3)^2(x + 5)^3$ The graph touches the x axis at $(3,0)$ and crosses at $(-5,0)$

Definition 11.18

If $(x - c)^k$ is a factor of a polynomial $P(x)$ and no higher power of $(x - c)$ is a factor of the polynomial, then c is called a **root of multiplicity** k of the polynomial P .

Example 11.19 (Multiplicity of a root)

Let $P(x) = (x - 1)^2(x - 2)(x + 4)^7(x + 7)^9$. List the roots of P and their multiplicities. Describe the behavior of the graph of P at each x -intercept.

Root	Multiplicity	Behavior of graph
$x = 1$	2	touches the x axis
$x = 2$	1	crosses the x axis
$x = -4$	7	crosses the x axis
$x = -7$	9	crosses the x axis

Theorem 11.20

Let $P(x)$ be a polynomial and let c be a root of multiplicity k of P .

- If k is odd, then the graph of P crosses the x -axis at $(c, 0)$.
- If k is even, then the graph of P touches the x -axis at $(c, 0)$ but does not cross the x -axis at $(c, 0)$.

(P-5) The graph of a polynomial of degree n has **at most** $n - 1$ **local extrema**.

Definition 11.21

A **local extremum** is a local minimum or a local maximum. The plural of “local extremum” is “**local extrema**”

On a graph, a local minimum appears as the point at the bottom of a valley and a local maximum appears as the point at the top of a mountain. This does not mean that it is the lowest valley or the highest mountain. You are looking for all valleys and all mountains.

Example 11.22 (Local Extrema)

- Could the graph of a 7^{th} degree polynomial have 8 local extrema? **No. It can have at most 6 local extrema.**
- Could the graph of a 7^{th} degree polynomial have 5 local extrema? **No. A 7^{th} degree polynomial can have 0, 2, 4, or 6 local extrema.**
- Could the graph of a 7^{th} degree polynomial have 4 local extrema? **Yes.**

Example 11.23 (Local Extrema)

How many local extrema could the graph of a polynomial of degree 10 have? **A polynomial of degree 10 can have 1, 3, 5, 7, or 9 local extrema.**

11.2.2 Polynomial Division

Example 11.24 (Review of Long Division)

Use long division to find the quotient and the remainder.

$$\begin{array}{r} 264 \\ 3 \overline{) 794} \\ \underline{600} \\ 194 \\ \underline{180} \\ 14 \\ \underline{12} \\ 2 \end{array}$$

The quotient is 264 and the remainder is 2.

Theorem 11.25 (The Division Algorithm)

Let $P(x)$ and $D(x)$ be polynomials. Then there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x)$ is the zero polynomial or the degree of $R(x)$ is less than the degree of $D(x)$.

Definition 11.26

In the Division Algorithm:

- $P(x)$ is the **dividend**.
- $D(x)$ is the **divisor**.
- $Q(x)$ is the **quotient**.
- $R(x)$ is the **remainder**.

Example 11.27 (Polynomial Division)

Find the quotient and the remainder.

$$\frac{3x^3 - 2x^2 + 4x - 3}{x + 4}$$

$$\begin{array}{r} 3x^2 - 14x + 60 \\ x + 4 \overline{) 3x^3 - 2x^2 + 4x - 3} \\ \underline{- 3x^3 - 12x^2} \\ - 14x^2 + 4x \\ \underline{14x^2 + 56x} \\ 60x - 3 \\ \underline{- 60x - 240} \\ - 243 \end{array}$$

The quotient $Q(x) = 3x^2 - 14x + 60$ and the remainder $R(x) = -243$

The next theorem includes the phrase, “The following are equivalent. . .” This means that all the statements are true or all of them are false. It is never the case that some are true and some are false.

Theorem 11.34

The following are equivalent for the polynomial $P(x)$:

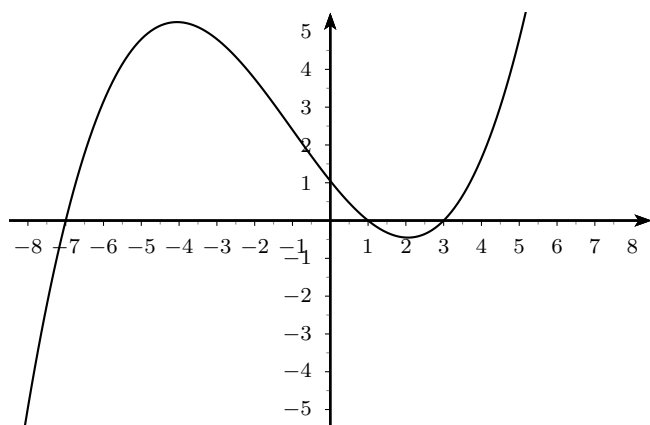
- $(x - c)$ is a factor of $P(x)$.
- $x = c$ is a root (or zero) of $P(x)$.
- $P(c) = 0$.
- $x = c$ is an x -intercept of the graph of P .

Theorem 11.35 (Number of Roots)

A polynomial of degree n has at most n distinct roots.

Example 11.36

Below is the graph of $y = f(x)$ and $f(x)$ is a polynomial function.



What is the remainder when $f(x)$ is divided by $(x + 7)$?

Since $f(-7) = 0$, then the remainder of $\frac{f(x)}{x + 7}$ is 0.

11.2.3 Finding Roots of Polynomials

The roots or zeros of a polynomial are often important in applications. When a polynomial has integer coefficients, the Rational Roots Theorem allows us to narrow the search for roots which are rational numbers.

Theorem 11.37 (The Rational Roots Theorem)

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (a_n \neq 0)$$

be a polynomial with integer coefficients. If $\frac{r}{s}$ is a rational number in lowest terms and $\frac{r}{s}$ is a root of P then

- r is a factor of a_0 AND
- s is a factor of a_n .

Example 11.38 (The Rational Roots Theorem)

Let $P(x) = 3x^3 - 8x^2 - x + 10$. List all of the possible rational roots of $P(x)$ as given by the Rational Roots Theorem. (Do not check to see which are actually zeros.)

The factors of $a_n = 3$ are ± 1 and ± 3 . The factors of $a_0 = 10$ are $\pm 1, \pm 2, \pm 5, \pm 10$. Thus, the possible rational roots of $P(x)$ are:

$$\pm \frac{1}{1}, \pm \frac{1}{3}, \pm \frac{2}{1}, \pm \frac{2}{3}, \pm \frac{5}{1}, \pm \frac{5}{3}, \pm \frac{10}{1}, \pm \frac{10}{3}$$

Example 11.39 (Finding all Rational Roots of a Polynomial)

Let $P(x) = 2x^4 + 8x^3 + 2x^2 - 16x - 12$. Find all the rational roots of $P(x)$. What does this tell you about the factors of $P(x)$? (*Hint: This may seem like a long list of values at which to evaluate P , but you can use the table function on your calculator to expedite the process.*)

The factors of $a_n = 2$ are $\pm 1, \pm 2$. The factors of $a_0 = -12$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. So, the possible rational roots of $P(x)$ are

$$\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{2}{1}, \pm \frac{2}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}, \pm \frac{4}{1}, \pm \frac{4}{2}, \pm \frac{6}{1}, \pm \frac{6}{2}, \pm \frac{12}{1}, \pm \frac{12}{2}$$

Using a calculator to check, we find that -3 and -1 are the rational roots of P . This means that two factors of P are $x + 3$ and $x + 1$.

Example 11.40

Completely factor 6160.

$$\begin{aligned}
 6160 &= 2 \cdot 3080 = 2 \cdot 2 \cdot 1540 = 2 \cdot 2 \cdot 2 \cdot 770 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 385 \\
 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 77 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \\
 &= 2^4 \cdot 5 \cdot 7 \cdot 11
 \end{aligned}$$

Suppose you know that $(x - c)$ is a factor of $P(x)$. What does this mean? How can you use this information to completely factor $P(x)$?

This means that the remainder of $\frac{P(x)}{x - c}$ is zero and $P(x) = Q(x) \cdot (x - c)$. Now we can try to factor $Q(x)$ to find a complete factorization of $P(x)$.

Example 11.41 (Finding all Real Roots of a Polynomial)

Let $P(x) = 2x^4 + 8x^3 + 2x^2 - 16x - 12$. Find all the real roots of $P(x)$.

We found earlier that $x = -3$ and $x = -1$ are the rational roots of $P(x)$. That means $P(x) = Q(x)(x + 3)(x + 1)$ where $Q(x)$ is the quotient of $\frac{P(x)}{(x + 3)(x + 1)} = \frac{P(x)}{x^2 + 4x + 3}$. We can use long division to find $Q(x)$.

$$\begin{array}{r}
 2x^2 - 4 \\
 \underline{x^2 + 4x + 3 2x^4 + 8x^3 + 2x^2 - 16x - 12} \\
 - 2x^4 - 8x^3 - 6x^2 \\
 \underline{ - 2x^4 - 8x^3 - 6x^2} \\
 - 4x^2 - 16x - 12 \\
 \underline{ - 4x^2 - 16x - 12} \\
 4x^2 + 16x + 12 \\
 \underline{ 4x^2 + 16x + 12} \\
 0
 \end{array}$$

Thus, $Q(x) = 2x^2 - 4$. Since $P(x) = Q(x)(x + 3)(x + 1)$, then the remaining roots of $P(x)$ are also roots of $Q(x)$. To find the roots of $Q(x)$, we can solve $Q(x) = 0$ for x .

$$\begin{aligned}
 2x^2 - 4 &= 0 \\
 2x^2 &= 4 \\
 x^2 &= 2 \\
 x &= \pm\sqrt{2}
 \end{aligned}$$

Thus, the real roots of $P(x)$ are $x = -3, -1, -\sqrt{2}, \sqrt{2}$.