Worksheet # 1: Review

1. (MA 113 Exam 1, Problem 1, Spring 2007). Find the equation of the line that passes through $(1, 2)$ and is parallel to the line $4x + 2y = 11$. Put your answer in $y = mx + b$ form.

2. Find the slope, x-intercept, and y-intercept of the line $3x - 2y = 4$.

3. Write the equation of the line through $(2, 1)$ and $(-1, 3)$ in point slope form.

4. Write the equation of the line containing $(0, 1)$ and perpendicular to the line through $(0, 1)$ and $(2, 6)$.

5. The quadratic polynomial $f(x) = x^2 + bx + c$ has roots at -3 and 1. What are the values of b and c?

6. Let $f(x) = Ax^2 + Bx + C$. If $f(1) = 3$, $f(-1) = 7$, and $f(0) = 4$ what are the values of A, B and C?

7. Find the intersection of the lines $y = 5x + 10$ and $y = -8x - 3$. Remember that an intersection is a point in the plane, hence an ordered pair.

8. Recall the definition of the absolute value function:

$$|x| = \begin{cases} x & x \geq 0 \\ \\ -x & x < 0 \end{cases}.$$

   Sketch the graph of this function. Also, sketch the graphs of the functions $|x + 4|$ and $|x| + 4$.

9. A ball is thrown in the air from ground level. The height of the ball in meters at time t seconds is given by the function $h(t) = -4.9t^2 + 30t$. At what time does the ball hit the ground? Units!

10. True or false?

    (a) For any function f, $f(s + t) = f(s) + f(t)$.
    (b) If $f(s) = f(t)$ then $s = t$.
    (c) A circle can be the graph of a function.
    (d) A function is a rule which assigns exactly one output $f(x)$ to every input x.
    (e) If $f(x)$ is increasing then $f(-52.55) \leq f(1752.0001)$.

Worksheet # 2: Functions and Inverse Functions; Logarithms

1. (MA 113 Exam I, Problem 2, Spring 2009). Consider the function $f(x) = \dfrac{4x+1}{3x-2}$. Determine the inverse function of f.

2. Let $f(x) = x^3 + 1$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and specify their domains.

3. Suppose the graph of $f(x)$ is given. Write an equation for the graph obtained by first shifting the graph of $f(x)$ up 3 units and left by 2 units, and then compressing the resulting graph horizontally by a factor of 10.

4. Suppose the graph of $g(x)$ is given by the equation $g(x) = f(2x - 5) + 7$. In terms of standard transformations describe how to obtain $g(x)$ from the graph of $f(x)$.

5. Find the domain and range of the following functions.

   (a) $f(x) = 15$
   (b) $f(x) = \sqrt{x^2 + 2x + 1}$
   (c) $f(x) = \sqrt{x^2 - 2x - 3}$
   (d) $f(x) = \dfrac{x}{|x|}$

6. Profit is the difference between total revenues and total costs. Suppose that Company W produces good Y. Let x denote the quantity of good Y sold. Suppose

$$R(x) = 15x$$

   and

$$C(x) = \frac{1}{10}x^2 + x + 30$$

   are the company's revenue and cost functions respectively for sales of this good.

   (a) Find the company's profit function $P(x)$.
   (b) Company W would really like to know how much of good Y they must sell to break even. Find the quantity x of good Y that the company must sell to make neither a profit nor a loss.

7. Compute each of the following logarithms exactly. Do not use a calculator.

   (a) $\log_{10} \sqrt{10^3}$
   (b) $\log_3(1/27)$
   (c) $\log_2 6 - \log_2 15 + \log_2 20$
   (d) $\log_{10}(\log_{10}(\log_{10}(10^{10^{100}})))$

8. Express each of the following as a single logarithm.

   (a) $\log_{10}(5) - \log_{10}(3) + \log_{10}(2)$
   (b) $\log_3(a + b) - 15\log_3(c) + 17\log_3(d)$

9. Solve the following equations for x

   (a) $10^{2x+1} - 7 = 0$
   (b) $\log_2(x) + \log_2(x - 1) = 1$
   (c) $3^{ax} = C \cdot 3^{bx}$, $a \neq b$.

10. True or false?

   (a) Every function has an inverse.
   (b) Every function will pass the vertical line test.
   (c) Every function will pass the horizontal line test.
   (d) $f \circ g(x) = g \circ f(x)$.
   (e) There is a function whose graph is an oval.
   (f) No function can be both even and odd.

Worksheet # 3: Tangents and Velocity

1. Sketch the graphs of the following functions using your knowledge of basic functions and transformations. Then sketch the tangent line to the curve at the specified point.

   (a) $f(x) = x^2 + 1$, $x = 2$

   (b) $f(x) = -|x| + 3$, $x = -1$

   (c) $f(x) = (x-2)^3 - 1$, $x = 2$

   (d) $f(x) = 2^{x-1} + 1$, $x = 1$.

2. (Adapted from MA 113 Exam I, Problem 6, Spring 2009). A particle is moving along a straight line so that its position at time t seconds is given by $s(t) = 4t^2 - t$.

   (a) Find the average velocity of the particle over the time interval $[1, 2]$.

   (b) Determine the average velocity of the particle over the time interval $[2, t]$ where $t > 2$. Simplify your answer. [Hint: Factor the numerator.]

   (c) Based on your answer in (b) can you guess a value for the instantaneous velocity of the particle at $t = 2$?

3. Let $x(t)$ be the function which describes the position of a particle traveling along the x-axis. Suppose the point $(15, 6)$ is on the graph of $x(t)$ and the tangent line at this point is given by $y = -3$. At time $t = 15$, determine the particle's position and instantaneous velocity.

4. (Problem 4, p. 87 in the text.) The point $P(3, 1)$ lies on the curve $y = \sqrt{x-2}$.

   (a) If Q is the point $(x, \sqrt{x-2})$, find a formula for the slope of the secant line PQ.

   (b) Using your formula from part (a) and a calculator, find the slope of the secant line PQ for the following values of x. [1] Keep 4 decimal places of accuracy and be careful with rounding.

      i. 2.9

      ii. 2.99

      iii. 2.999

      iv. 3.1

      v. 3.01

      vi. 3.001

   (c) Using the results of part (b), guess the value of the slope of the tangent line to the curve at $P(3, 1)$.

   (d) Using the slope from part (c), find the equation of the tangent line to the curve at $P(3, 1)$.

5. (Adapted from problem 5, p. 87 in the text.) If a ball is thrown in the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $f(t) = 40t - 16t^2$.

   (a) Using a calculator, find the average velocity of the ball for the time period beginning when $t = 2$ and lasting

      i. 0.5 second

      ii. 0.1 second

      iii. 0.05 second

      iv. 0.01 second

   (b) Estimate the instantaneous velocity when $t = 2$.

   (c) Find a general formula for the average velocity of the ball for the time period beginning at t and lasting h seconds. Simplify your answer.

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[1] TI-8X calculator tip: Hit the "y=" button and put your formula from part a.) in, say, the y_1 position. Then go to the home screen, access the y-vars menu, and use it to type $y_1(x)$ to find the value of y_1 at the point x. You could also use the table feature.

(d) Based on your answer in (c), can you guess a general formula for the instantaneous velocity at time t? [Hint: What does the result in (c) look like as h gets very close to 0?]

6. Let $s(t)$ describe the position of a particle traveling along the x-axis at time t. Let $v(t)$ be the particle's instantaneous velocity and $a(t)$ be the instantaneous acceleration function at time t. Determine if the following statements are true or false.

   (a) If $v(t) = 0$ then the particle is at rest at time t.

   (b) If $s(t) = 0$ then the particle is at the origin at time t.

   (c) If $a(t) > 0$ then the particle must be speeding up at time t.

   (d) If $a(t) = 0$ and $s(t) = 0$, the particle will remain at the origin.

   (e) If $a(t) > 0$ and $v(t) = 0$ at time t, the particle will soon begin traveling to the right.

   (f) If $v(t)$ is constant for all t, then $a(t) = 0$.

   (g) Suppose $v(t) > 0$ and $s(t) > 0$ for all time values. Then the particle will stay to the right of the origin forever.

Worksheet # 4: Introduction to Limits

1. Comprehension check.

   (a) In words, describe what
   $$\lim_{x \to a} f(x) = L$$
   means.

   (b) How can one-sided limits help you to determine if a limit exists?

   (c) In words, what does
   $$\lim_{x \to a} f(x) = \infty$$
   mean?

   (d) Suppose $\lim_{x \to 1} f(x) = 2$. Does $f(1) = 2$?

   (e) Suppose $f(1) = 2$. What can be said about $\lim_{x \to 1} f(x)$?

2. Let
   $$f(x) = \begin{cases} x^2, & x \leq 0 \\ x - 1, & 0 < x,\ x \neq 2 \\ -3, & x = 2 \end{cases}.$$

   (a) Sketch the graph of f.

   (b) Compute the following.

      i. $\lim_{x \to 0^-} f(x)$

      ii. $\lim_{x \to 0^+} f(x)$

      iii. $\lim_{x \to 0} f(x)$

      iv. $f(0)$

      v. $\lim_{x \to 2^-} f(x)$

      vi. $\lim_{x \to 2^+} f(x)$

      vii. $\lim_{x \to 2} f(x)$

      viii. $f(2)$

3. In the following, sketch the functions and use the sketch to compute the limit.

   (a) $\lim_{x \to 3} \pi$

   (b) $\lim_{x \to \pi} x$

   (c) $\lim_{x \to a} |x|$

   (d) $\lim_{x \to 3} 2^x$

4. Compute the following limits or explain why they fail to exist:

   (a) $\lim_{x \to -3^+} \dfrac{x+2}{x+3}$

   (b) $\lim_{x \to -3^-} \dfrac{x+2}{x+3}$

   (c) $\lim_{x \to -3} \dfrac{x+2}{x+3}$

   (d) $\lim_{x \to 0^-} \dfrac{1}{x^3}$

5. (Problem 40, p. 99 in the text). In the theory of relativity, the mass of a particle with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

6. Let

$$f(x) = \begin{cases} 2x + 2, & x > -2 \\ a, & x = -2 \\ kx, & x < -2 \end{cases}.$$

Find k and a so that $\lim_{x \to -2} f(x) = f(-2)$.

Worksheet # 5: Limit Laws

1. Given $\lim_{x \to 2} f(x) = 5$ and $\lim_{x \to 2} g(x) = 2$, use limit laws (justify your work) to compute the following limits. Note when working through a limit problem that your answers should be a chain of equalities. Make sure to keep the $\lim_{x \to a}$ operator until the very last step.

   (a) $\lim_{x \to 2} 2f(x) - g(x)$.

   (b) $\lim_{x \to 2} \frac{f(x)g(x)}{x}$.

   (c) $\lim_{x \to 2} f(x)^2 + x \cdot g(x)^2$.

   (d) $\lim_{x \to 2} [f(x)]^{\frac{3}{2}}$.

2. Calculate the following limits if they exist or explain why the limit does not exist.

   (a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

   (b) $\lim_{x \to 1} \frac{x^2 - 1}{x - 2}$

   (c) $\lim_{x \to 2^+} \frac{x^2 - 1}{x - 2}$

   (d) $\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3}$

3. Find the value of c such that $\lim_{x \to 2} \frac{x^2 + 3x + c}{x - 2}$ exists. What is the limit?

4. Show that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist by examining one-sided limits. Then sketch the graph of $\frac{|h|}{h}$ and check your reasoning.

5. True or false?

   (a) The direct substitution property can always be used to compute limits.

   (b) Let $f(x) = \frac{(x+2)(x-1)}{x-1}$ and $g(x) = x + 2$. Then $f(x) = g(x)$.

   (c) Let $f(x) = \frac{(x+2)(x-1)}{x-1}$ and $g(x) = x + 2$. Then $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x)$.

   (d) If both the one-sided limits of $f(x)$ exist as x approaches a, then $\lim_{x \to a} f(x)$ exists.

   (e) Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0$ be a polynomial with coefficients $c_n, c_{n-1}, ..., c_0$. Then $\lim_{x \to a} p(x) = c_n a^n + c_{n-1} a^{n-1} + ... + c_1 a + c_0$.

   (f) If $\lim_{x \to a} f(x)$ exists then $\lim_{x \to a} f(x) = f(a)$.

Worksheet # 6: Continuity

1. Comprehension check.

   (a) State and explain the intermediate value theorem.

   (b) Define what it means for $f(x)$ to be continuous at the point $x = a$. What does it mean if $f(x)$ is continuous on the interval $[a, b]$? What does it mean to say $f(x)$ is continuous?

   (c) There are three distinct ways in which a function will fail to be continuous at a point $x = a$. Describe the three types of discontinuity. Provide a sketch and an example of each type.

   (d) True or false? Every function is continuous on its domain.

   (e) True or false? The sum, difference, and product of continuous functions are all continuous.

   (f) If $f(x)$ is continuous at $x = a$, what can you say about $\lim_{x\to a^+} f(x)$?

   (g) Suppose $f(x), g(x)$ are continuous everywhere. What is $\lim_{x\to a} \dfrac{f(x)g(x) - f(x)^3}{g(x)^2 + 1}$?

2. Using the definition of continuity and properties of limits, show that the following functions are continuous at the given point a.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \dfrac{x^2 + 3x + 1}{x + 3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2 - 9}$, $a = 4$.

3. Give the largest domain on which the following functions are continuous. Use interval notation.

   (a) $f(x) = \dfrac{x + 1}{x^2 + 4x + 3}$

   (b) $f(x) = \dfrac{x}{x^2 + 1}$

   (c) $f(x) = \sqrt{2x - 3} + x^2$

   (d) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ x + 1 & \text{if } 0 < x < 2 \\ -(x - 2)^2 & \text{if } x \geq 2 \end{cases}$

4. State the intermediate value theorem and use the theorem to find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

5. (Similar to MA 113 Exam I, problem 8, Spring 2009.) Let c be a number and consider the function

$$f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$$

   (a) Find all numbers c such that $\lim_{x\to 1} f(x)$ exists.

   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

Worksheet # 7: Derivatives

1. Comprehension check.

   (a) What does it mean for a function to be continuous at the point a? What does it mean for a function be differentiable at the point a?

   (b) Are differentiable functions also continuous? Are continuous functions also differentiable?

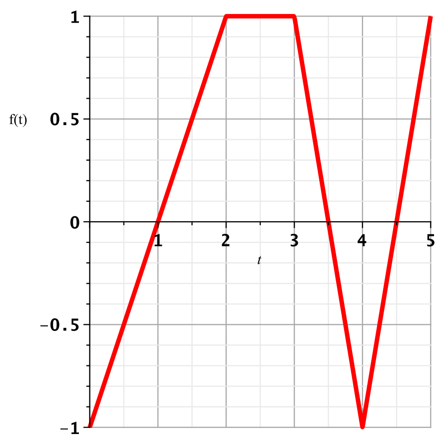
   (c) You have seen three ways in which a function can fail to be differentiable at a point. Sketch these three cases.

   (d) The tangent line to the graph of $g(x)$ at $x = 1$ is given by $y = 5x + 1$. Find $g(1)$ and $g'(1)$.

   (e) Give the two formulas for the definition of the derivative of a function $f(x)$ at a point a.

   (f) What does the derivative of $f(x)$ at $x = a$ describe?

2. A particle is traveling along the x-axis. Below is a graph of its position function $f(t)$ for the time interval $[0, 5]$.



   (a) Graph the particle's velocity function on the time interval $[0, 5]$.

   (b) Graph the particle's acceleration function on the time interval $[0, 5]$.

   (c) For what time intervals is the particle traveling left? Right? When is it at rest?

3. Find $f'(a)$ using either formula of the definition for the derivative:

   (a) $f(x) = 3x^2 - 2x + 1$

   (b) $f(x) = \dfrac{1}{x+3}$

   (c) $f(x) = \sqrt{x}$

4. Use 2(c) to find the tangent line to $f(x) = \sqrt{x}$ when $x = 4$.

5. Let

$$h(t) = \begin{cases} at + b & t \le 0 \\ t^3 + 1 & t > 0 \end{cases}$$

   Find a and b so that h is differentiable at $t = 0$.

Worksheet # 8: Review for Exam I

1. Calculate the following limits using the limit laws. Carefully show your work and use only one limit law per step.

   (a) $\lim_{x\to 0}(2x-1)$

   (b) $\lim_{x\to -1}\frac{x^2+1}{x}$

   (c) $\lim_{x\to 1}(3x^3-2x^2+4)$

2. (a) State the Intermediate Value Theorem.

   (b) Use the Intermediate Value Theorem to show that the polynomial $f(x)=x^3+2x-1$ has a zero in some interval of length 1.

   (c) Prove that you were once π feet tall.

3. Use the definition of the derivative to find $f'(x)$. Do not use the derivative laws if you know them, because you will not be able to use them on the exam.

   (a) $f(x)=\frac{1}{x}$

   (b) $f(x)=3x^2+2$

4. (a) State the definition of continuity of a function $f(x)$ at $x=a$

   (b) Find the constant a so that the function is continuous on the entire real line.

$$f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a} & \text{if } x\neq a\\ 8 & \text{if } x=a\end{cases}$$

5. Let $f(x)=|x|$. From the definitions, prove that $f(x)$ is continuous at $x=0$ but not differentiable there. Explain how you could surmise this fact from the graph of $f(x)$.

6. The line tangent to the graph of $f(x)$ at $x=3$ is $y=-2x+1$. Using this fact, find $f(3)$ and $f'(3)$.

Worksheet # 9: Derivatives of Polynomial and Exponential Functions

1. Comprehension check.

   (a) True or false: If $f'(x) = g'(x)$ then $f(x) = g(x)$?

   (b) Find an example which shows that in general $(f(x)g(x))' \neq f'(x)g'(x)$.

   (c) Suppose $f'(a)$ exists. Does $\lim_{x \to a} f(x) = f(a)$? Explain.

   (d) How is the number e defined?

2. Compute the derivative of the following functions.

   (a) $f(x) = \frac{9}{4}x^8$

   (b) $k(x) = 3e^x + x^2 + 1$

   (c) $k(x) = \frac{A}{x^4} + Bx^2 + Cx + D$

   (d) $n(x) = e^{x+2} + 1$

   (e) $l(x) = \left(x + \frac{1}{x}\right)^2$

   (f) $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$

3. Let $f(x) = x^2 + 3x - 5$. Where is the slope of $f(x)$ positive? Negative? Zero?

4. Find an equation for the tangent line to $y = x^{3/2} + 2$ at $x = 3$.

5. (MA 113 Exam II, problem 8, Spring 09). Consider the function $f(x) = x^2 - 2x + 2$. Find the equations of the tangent lines to this parabola which pass through the point $(3, 4)$. As usual, a sketch of the curve and the tangent lines should be your first step in solving the problem.

6. Suppose $x(t) = 2t^3 + 3t^2 - 72t + 50$ gives the position of a particle on the x axis at time t. Determine all time values when the particle is at rest.

Worksheet # 10: Product and Quotient Rules

1. Show by way of example that, in general,

$$\frac{d(f \cdot g)}{dx} \neq \frac{df}{dx} \cdot \frac{dg}{dx}$$

and

$$\frac{d\left(\frac{f}{g}\right)}{dx} \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}.$$

2. State the quotient and product rule and be sure to include all necessary hypotheses.

3. Compute the first derivative of each of the following:

   (a) $f(x) = \dfrac{\sqrt{x}}{x-1}$

   (b) $f(x) = (3x^2 + x)e^x$

   (c) $f(x) = \dfrac{e^x}{2x^3}$

   (d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x-1}{\sqrt{x}}\right)$

   (e) $f(x) = \dfrac{2x}{4+x^2}$

   (f) $f(x) = \dfrac{ax+b}{cx+d}$

   (g) $f(x) = \dfrac{(x^2+1)(x^3+2)}{x^5}$

4. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the n-th derivative of f.

5. Find an equation of the tangent line to the given curve at the specified point.

   (a) $y = x^2 + \dfrac{e^x}{x^2+1}$ at the point $x = 3$

   (b) $y = 2xe^x$, $x = 0$

6. Suppose that $f(2) = 3$, $g(2) = 2$, $f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

   (a) $h(x) = 5f(x) + 2g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   (d) $h(x) = \dfrac{g(x)}{1+f(x)}$

Worksheet # 11: Trigonometric Functions

1. Convert the angle $\pi/12$ to degrees and the angle $900°$ to radians.

2. Find the exact values of the following expressions. Do not use a calculator.

   (a) $\arctan(1)$

   (b) $\tan(\arctan(10))$

   (c) $\arcsin(\sin(7\pi/3))$

3. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

   (a) $\sqrt{3}\cos x + 2\tan x \cos^2 x = 0$

   (b) $3\cot^2(x) = 1$

   (c) $2\cos x + \sin 2x = 0$

4. If $\sin(x) = \dfrac{2}{5}$ and $\sec(x) = \dfrac{-5}{3}$, find $\csc(x), \cot(x), \cos(x), \tan(x), \sin(2x)$.

5. Find the length of the circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

6. A clock lies in the coordinate plane so that its center is at the origin. The hour hand is 5 cm long and the minute hand is 15 cm long. Find the coordinates of the tips of each hand at $3:50$ pm.

7. Differentiate each of the following functions:

   (a) $f(t) = \cos(t)$

   (b) $g(u) = \dfrac{1}{\cos(u)}$

   (c) $r(\theta) = \theta^3 \sin(\theta)$

   (d) $s(t) = \tan(t) + \csc(t)$

   (e) $h(x) = \sin(x)\csc(x)$

   (f) $f(x) = x^2 \sin^2(x)$

8. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

   (a) Determine the particle's speed (speed=|velocity|) at π seconds.

   (b) Is the particle moving towards or away from the origin at π seconds. Explain.

   (c) Now, find the velocity of the particle at time $t = 3\pi/2$. Is the particle moving towards the origin or away from the origin?

Worksheet # 12: Chain Rule

1. (MA 113 Exam II, problem 9, Spring 2009).

   (a) Carefully state the chain rule. Use complete sentences.

   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = 1/4$, and $g'(2) = 2$. Find each of the following:

      i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.

      ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

2. Differentiate each of the following and simplify your answer.

   (a) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$

   (b) $g(t) = \tan(\sin t)$

   (c) $h(u) = \sec^2 u + \tan^2 u$

   (d) $f(x) = e^{(3x^2+x)}$

   (e) $g(x) = \sin(\sin(\sin x))$

3. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 e^{3x}$, $x = 2$

   (b) $f(x) = \sin x + \sin^2 x$, $x = 0$

4. If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

5. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	f(x)	g(x)	f'(x)	g'(x)
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

   Find: $h'(-1), h'(3)$ and $k'(2)$.

6. Find all x values so that $f(x) = 2\sin x + \sin^2 x$ has a horizontal tangent at x.

7. Comprehension check for derivatives of trigonometric functions.

   (a) True or false? If $f'(x) = -\sin(\theta)$ then $f(\theta) = \cos(\theta)$.

   (b) True or False? If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \frac{2}{3}$ then the hypotenuse of the triangle must have length 3.

   (c) Let $f(\theta) = \sin(\theta)$. Find $f^{(435)}(\theta)$.

   (d) Differentiate both sides of the identity

$$\tan x = \frac{\sin x}{\cos x}$$

   to obtain a new trigonometric identity.

Worksheet # 13: Implicit Differentiation

1. Find dy/dx by implicit differentiation.
   (a) $x^3 + y^3 = 1$
   (b) $e^y \cos x = 1 + \sin(xy)$
   (c) $y^2(2 - x) = x^3$

2. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.
   (a) $x^2 + y^2 = x + y - x^3$, $(0, 1)$
   (b) $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$

3. Find the derivative of each of the following.
   (a) $f(x) = \arctan \sqrt{x}$
   (b) $g(x) = \arcsin x^2$
   (c) $h(x) = \arccos(e^{2x})$
   (d) $f(x) = \ln(x^2 + 2)$
   (e) $f(x) = \ln(e^{2x} + 5e^x + 3)$
   (f) $f(x) = \ln(\cos(x))$

4. The equation $x^2 - xy + y^2 = 3$ represents a "rotated" ellipse, that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points where this ellipse crosses the x-axis and show that the tangents at these points are parallel.

5. Prove:
$$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}.$$
[Hint: Use the same technique from the proof for the derivative formula for $\sin^{-1}(x)$. Start by writing $y = \cos^{-1}(x)$ and obtain an expression which can be differentiated implicitly.]

Worksheet # 14: Growth, Decay, and Rates of Change

1. Comprehension check for exponential functions.

   (a) Explain the intuition behind the simple population growth model
   $$\frac{dP}{dt} = kP$$
   where k is a positive constant. Describe some situations where this model may break down.

   (b) What is the unique (only) function satisfying $f'(x) = f(x)$ and $f(0) = 1$?

   (c) Find a function $f(x)$ such that $f'(x) = 3f(x)$ and $f(0) = 15$.

2. An explorer brought two rabbits (male and female) to a small island. Based on 30 years of data, the Rabbit Research Group has concluded that the rabbit population on the island doubles every year. Set up the proportional growth rate population equation and use it to predict the number of rabbits for 10, 50, and 100 years. What might be wrong with using this model to predict population values for large values of t?

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

   (a) Find an expression for the number of bacteria after t hours.

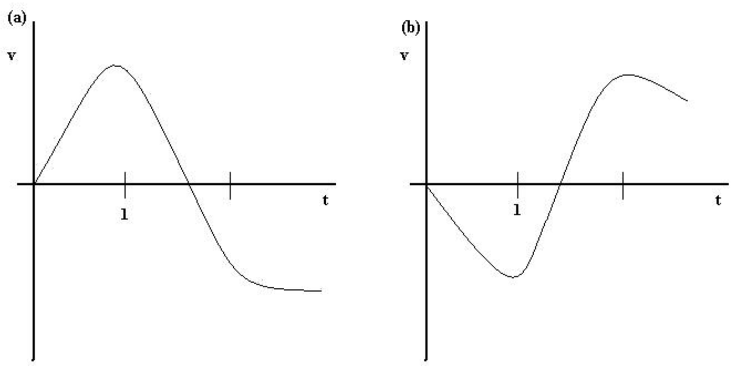
   (b) Find the number of bacteria after 3 hours.

   (c) Find the rate of growth after 3 hours.

   (d) When will the population reach 10,000 bacteria?

4. A sample of a chemical compound decayed to 95.47 % of its original mass after one year. What is the half life of the compound? How long would it take for the compound to decay to 20 % of its original mass.

5. Graphs of the **velocity** functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



6. A man uses a helium tank to inflate a large balloon. The balloon's surface area is given by $S = 4\pi r^2$ and its volume is given by $V = \frac{4}{3}\pi r^3$.

   (a) Find the rate of increase in surface area with respect to the radius when the diameter of the balloon is 2 ft.

   (b) Suppose the radius of the balloon at time t seconds is $r(t) = 2t + 1$. Find the rate of increase in surface area with respect to time when $t = 1$ sec.

   (c) Show that if the volume of the balloon is decreasing at a rate (with respect to time) proportional to its surface area, then the radius of the balloon is shrinking at a constant rate.

Worksheet # 15: Related Rates

1. A ladder of length 3 meters long is leaning against a wall. The base of the ladder is sliding away from the wall at a rate of .5 meters/second. Find the speed that the ladder is moving along the wall when the top of the ladder is 2 meters above the floor.

2. A person 5 ft tall walks along a straight path at a rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. Find the rate at which the person's shadow is changing for any time value.

3. A boat is being pulled towards a dock by a rope attached to the bow of the boat. The boat is approaching the dock at a rate of 3 meters/second. At the edge of the dock, the rope is one meter higher than it is at the bow of the boat. How fast is the rope being pulled in when the boat is 10 meters from the dock?

4. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

5. A baseball diamond is a square with side 90 ft. A player hits the ball and runs toward first base at a speed of 30 ft/s. How fast is his distance from second base decreasing when he is 2/3 of the way to first base?

Worksheet # 16: Review for Exam II

1. State:
   (a) The product rule and quotient rule.
   (b) The chain rule.
2. Compute the first derivative of each of the following functions.
   (a) $f(x) = \cos(4\pi x^3) + \sin(3x + 2)$
   (b) $b(x) = x^4 \cos(3x^2)$
   (c) $y(x) = e^{\sec 2\theta}$
   (d) $k(x) = \ln(7x^2 + \sin(x) + 1)$
   (e) $u(x) = (\arcsin 2x)^2$
   (f) $h(x) = \dfrac{8x^2 - 7x + 3}{\cos(2x)}$
   (g) $m(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$
   (h) $q(x) = \dfrac{e^x}{1 + x^2}$
   (i) $n(x) = \cos(\tan x)$
   (j) $w(x) = \arcsin x \cdot \arccos x$
3. The tangent line to $f(x)$ at $x = 3$ is given by $y = 2x - 4$. Find the tangent line to $g(x) = \dfrac{x}{f(x)}$ at $x = 3$. Put your answer in slope-intercept form.
4. (MA 113 Exam II, Problem 6, Fall 2008). Consider the curve $xy^3 + 12x^2 + y^2 = 24$. Assume this equation can be used to define y as a function of x (i.e. $y = y(x)$) near $(1, 2)$ with $y(1) = 2$. Find the equation of the tangent line to this curve at $(1, 2)$.
5. Let x be the angle in the interval $(-\pi/2, \pi/2)$ so that $\sin x = \dfrac{-3}{5}$. Find: $\sin(-x), \cos(x)$, and $\cot(x)$.
6. (Adapted from MA 113 Exam II, Problem 7, Fall 2008). The growth rate of the population in a bacteria colony at time t obeys the *differential equation*

$$P'(t) = kP(t)$$

   where k is a constant and t is measured in years.
   (a) Let A be a constant. Show that the function $P(t) = Ae^{kt}$ satisfies the differential equation.
   (b) If the colony initially has 100 bacteria and two years later has 200 bacteria, determine the values of A and k.
   (c) Suppose $P(t) = 100e^{.001t}$. When will the colony have 100,000 bacteria?
7. (MA 113 Exam II, Problem 9, Spring 2009). Suppose f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = 1/4$, and $g'(2) = 2$. Find:
   (a) $h'(2)$ where $h(x) = \ln([f(x)]^2)$;
   (b) $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.
8. (MA 113 Exam II, Problem 9, Spring 2007). Abby is north of Oakville and driving north along Road A. Boris is east of Oakville and driving west on Road B. At 11:57 AM, Boris is 5 km east of Oakville and traveling west at a speed of 60 kmph and Abby is 10 km north of Oakville and traveling north at a speed of 50 kmph.

(a) Make a sketch showing the location and direction of travel for Abby and Boris.

(b) Find the rate of change of the distance between Abby and Boris at 11:57 AM.

(c) At 11:57 AM, is the distance between Abby and Boris increasing, decreasing, or not changing?

9. (MA 113 Exam II, Problem 10, Fall 2008). The function $\arctan x$ is defined by $y = \arctan x$, if and only if $x = \tan y$, $-\pi/2 < y < \pi/2$. Use implicit differentiation to find the derivative of $\arctan x$. [Hint: use a trigonometric identity.]

Worksheet # 17: Maxima and Minima

1. Define the following terms:
   - Critical number.
   - Local maximum.
   - Absolute maximum.

2. Sketch:

   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.

   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.

   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.

   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.

   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.

3. Find the critical numbers for the following functions

   (a) $f(x) = x^3 + x^2 + 1$

   (b) $f(x) = \dfrac{x}{x^2 + 3}$

   (c) $f(x) = |5x - 1|$

4. Given a continuous function on a closed interval $[a, b]$, carefully describe the method you would use to find the absolute minimum and maximum value of the function.

5. Use the extreme value theorem to find the absolute maximum and absolute minimum value of the following function on the given intervals. Specify the values where these extrema occur.

   (a) (MA 113 Exam III, Problem 2, Fall 2008). $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (b) (MA 113 Exam III, Problem 2, Spring 2009). $f(t) = t + \sqrt{1 - t^2}$, $[-1, 1]$.

6. Comprehension check.

   (a) True or false? An absolute maximum is always a local maximum.

   (b) True or false? If $f'(c) = 0$ then f has a local maximum or local minimum at c.

   (c) True or false? If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.

   (d) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.

Worksheet # 18: The Mean Value Theorem

1. State the mean value theorem and illustrate the theorem in a sketch.

2. (MA 113 Exam III, Problem 8(c), Spring 2009). Suppose that g is differentiable for all x and that $-5 \leq g'(x) \leq 2$ for all x. Assume also that $g(0) = 2$. Based on this information, is it possible that $g(2) = 8$?

3. Section 4.2 in the text contains the following important corollary which you should commit to memory:

   > **Corollary 7, p. 284:** *If $f'(x) = g'(x)$ for all x in an interval (a, b) then $f(x) = g(x) + c$ for some constant c.*

   Use this result to answer the following questions:

   (a) If $f'(x) = \sin(x)$ and $f(0) = 15$ what is $f(x)$?

   (b) If $f'(x) = \sqrt{x}$ and $f(4) = 5$ what is $f(x)$?

   (c) If $f'(x) = k$ where k is a constant, show that $f(x) = kx + d$ for some other constant d.

4. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   (a) $f(x) = e^{-2x}$, [0,3]

   (b) $f(x) = \dfrac{x}{x+2}$, [1,4]

5. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

6. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

7. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ -x^2 + 3x + a & \text{if } 0 < x < 1 \\ mx + b & \text{if } 1 \leq x \leq 2 \end{cases}$$

   satisfy the hypotheses of the Mean Value Theorem on the interval [0,2]?

8. Determine whether the following statements are true or false. If the statement is false, provide a counterexample.

   (a) If f is differentiable on the open interval (a, b), $f(a) = 1$, and $f(b) = 1$, then $f'(c) = 0$ for some c in (a, b).

   (b) If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.

   (c) Suppose f is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then $f'(\frac{a+b}{2}) = 0$.

   (d) If f is differentiable everywhere and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.

10-28-2009

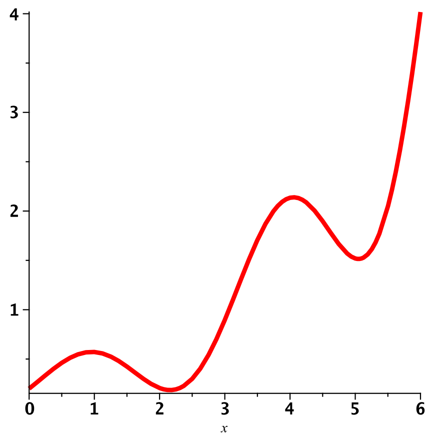
Worksheet # 19: Asymptotes and Curve Sketching

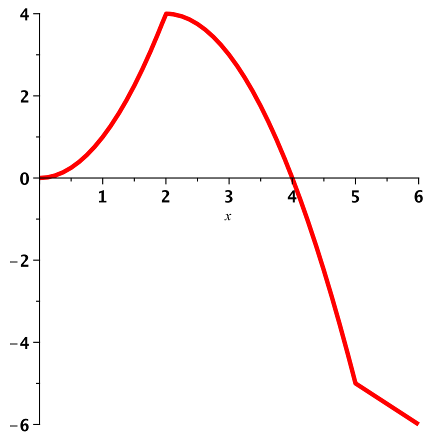
1. (a) Define the terms **horizontal asymptote** and **vertical asymptote**.
   (b) Explain the difference between $\lim_{x\to -3} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -3$.
   (c) Explain what $\lim_{x\to\infty} f(x) = 150$ means.
   (d) Explain what $\lim_{x\to 150} f(x) = 150$ means.
   (e) Explain how to use the first derivative test to identify and classify local extrema of the differentiable function $f(x)$.
   (f) Explain how to use the second derivative test to identify and classify local extrema of the twice differentiable function $f''(x)$. Does the test always work? What should you do if it fails?

2. (MA 113 Exam III, Problem 1, Spring 2009). Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 4$ on $(-\infty, \infty)$.
   (a) Find the critical point(s) of f.
   (b) Find the intervals of increase and decrease for f.
   (c) Find the local extrema of f.

3. (MA 113 Exam III, Problem 3, Spring 2009). Consider the function $f(x) = 2x + \sin x$ on $(-\pi, 2\pi)$.
   (a) Find the interval(s) of concavity of the graph of $f(x)$; show your work.
   (b) Find the point(s) of inflection of the graph of $f(x)$; justify your work.

4. For each graph of the function f:





   (a) Find the open interval(s) where f is increasing.
   (b) Find the open interval(s) where f is decreasing.
   (c) Find the open interval(s) where f is concave up.
   (d) Find the open interval(s) where f is concave down.
   (e) Identify all points of inflection.
   (f) Identify and classify all local extrema on $[0, 6]$.

5. Find the local maximum and minimum values of $f(x) = \dfrac{x}{x^2+4}$ using the first derivative test.

6. Find the local maximum and minimum values of $f(x) = x^5 - 5x + 4$ using the second derivative test.

7. Sketch the graph of a function f with all of the following properties.
   - $\lim_{t\to\infty} f(t) = 2$
   - $\lim_{t\to -\infty} f(t) = 0$

- $\lim_{t \to 0^+} f(t) = \infty$
- $\lim_{t \to 0^-} f(t) = -\infty$
- $\lim_{t \to 4} f(t) = 3$
- $f(4) = 6$

8. Evaluate the following limits, if they exist. If a limit does not exist, explain why.

   (a) $\lim_{t \to \infty} \frac{3t^2 - 7t}{t - 8}$

   (b) $\lim_{t \to \infty} \frac{2t^2 - 6}{t^4 - 8t + 9}$

   (c) $\lim_{t \to -\infty} \frac{t}{t^6 - 4t^2}$

   (d) $\lim_{t \to -\infty} 3$

   (e) $\lim_{t \to \pm\infty} \frac{5t^3 - 7t^2 + 9}{t^2 - 8t^3 - 8999}$

   (f) $\lim_{u \to \infty} \sqrt{16u^2 - u} - 4u$

Worksheet # 20: L'Hospital's Rule and Curve Sketching

1. Carefully, state L'Hospital's Rule.

2. Compute the following limits. Use l'Hospital's Rule where appropriate but first check that no easier method will solve the problem.

   (a) $\lim_{x\to 1} \frac{x^9 - 1}{x^5 - 1}$

   (b) $\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$

   (c) $\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$

   (d) $\lim_{x\to \infty} \frac{e^x}{x^3}$

   (e) $\lim_{x\to -\infty} x^2 e^x$

   (f) $\lim_{x\to \infty} x^3 e^{-x^2}$

3. Choose a and b so that
$$\lim_{x\to 0} \frac{\sin 3x + ax + bx^3}{x^3} = 0.$$

4. (MA 113 Exam III, Problem 11, Spring 2008). Sketch the graph of a function $f(x)$ defined for $x > 0$ such that

   (a) $\lim_{x\to 0^+} f(x) = 3$,

   (b) $f(2) = f(4) = 2, f(3) = 4$,

   (c) $\lim_{x\to\infty} f(x) = f(1) = 1$,

   (d) $f''(x)$ exists and is continuous for all $x > 0$,

   (e) $f'(1) = f'(3) = f''(2) = f''(4) = 0$, and $f'(x)$ and $f''(x)$ are not zero for all other values of x.

5. Sketch the graph of a function which satisfies all of the following properties.

   - $f(1) = f'(1) = 0$
   - $\lim_{x\to 2^+} f(x) = \infty$
   - $\lim_{x\to 2^-} f(x) = -\infty$
   - $\lim_{x\to 0} f(x) = -\infty$
   - $\lim_{x\to -\infty} f(x) = \infty$
   - $\lim_{x\to \infty} f(x) = 0$
   - $f''(x) > 0$ when $x > 2$
   - $f''(x) < 0$ when $x < 0$ and $0 < x < 2$.

6. (a) Outline a procedure for sketching the curve $y = f(x)$ using the tools of calculus.

   (b) Sketch the following curves using the procedure you described above. Check your answers with a calculator.

      i. $y = 8x^2 - x^4$

      ii. $y = \frac{x}{x^2 - 1}$

Worksheet # 21: Optimization

1. (MA 113 Exam III, Problem 10, Spring 2009). Find the point(s) on the hyperbola $y = \frac{16}{x}$ that is (are) closest to $(0, 0)$. Be sure to clearly state what function you choose to minimize or maximize and why.

2. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

3. (Problem 59, p. 349). A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1,000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

4. An oil company needs to run a pipeline to a nearby station. The station and oil company are on opposite sides of a river that is 1 km wide and that runs exactly west-east. Also, the station is 10 km east of where the oil company would be if it was on the same side of the river. The cost of land pipe is 200 dollars per meter and the cost of water pipe is 300 dollars per meter. Set up an equation whose solution(s) are the critical points for the cost function for this problem.

5. A 10 ft length of rope is to be cut into two pieces to form a square and a circle. How should the rope be cut to maximize the enclosed area?

6. (MA 113 Exam III, Problem 10, Spring 2007). Consider a can in the shape of a right circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72 π cubic centimeters and whose cost is as small as possible.

   (a) Find a function $f(r)$ which gives the cost of the can in terms of radius r. Be sure to specify the domain.

   (b) Give the radius and height of the can with least cost.

   (c) Explain how you known you have found the can of least cost.

Worksheet # 22: Linear Approximation

1. For each of the following, use a linear approximation to estimate the actual value.

   (a) $\tan(44°)$

   (b) $(3.01)^3$

   (c) $\sqrt{17}$

   (d) $8.06^{2/3}$

2. Suppose we want to paint a sphere of radius 200 cm with a coat of paint .2 cm thick. Use a linearization to approximate the amount of paint we need to do the job.

3. (MA 113 Exam III, Problem 4, Spring 2009). Let $f(x) = \sqrt{16 + x}$. First, find the linear approximation to $f(x)$ at $x = 0$. Then use the linear approximation to estimate $\sqrt{15.75}$. Present your solution as a rational number (fraction).

4. Your physics professor tells you that you can replace $\sin\theta$ with θ in equations when θ is close to zero. Explain why this is reasonable.

5. Suppose we measure the radius of a sphere as 10 cm with an accuracy of $\pm$ .5 cm. Use a linear approximation to estimate the maximum error in (a) the computed surface area and (b) the computed volume.

Worksheet # 23: Antiderivatives

1. Find the most general antiderivative for each of the following functions.

   (a) $x - 3$

   (b) $\frac{1}{4}x^6 - 5x^3 + 9x$

   (c) $(x+1)(9x-8)$

   (d) $\sqrt{x} - \frac{2}{\sqrt{x}}$

   (e) $\frac{5}{x}$

   (f) $\sqrt{x^5} - 40$

   (g) $\frac{x^3 - 8x^2 + 5}{x^2}$

   (h) $\frac{5}{x^6}$

   (i) $\frac{\sqrt{x}}{x^2} + \frac{3}{4}x^3$

   (j) $\frac{2}{5}x^e$

   (k) $\frac{1}{x-3}$

   (l) $\sin(\theta) - \sec^2(\theta)$

2. Find the values of the parameter A and B so that

   (a) $F(x) = (Ax + B)e^x$ is an antiderivative of $f(x) = xe^x$.

   (b) $H(x) = e^{2x}(A\cos x + B\sin x)$ is an antiderivative of $h(x) = e^{2x}\sin x$.

3. A particle moves along a straight line so that its velocity is given by $v(t) = t^2$. What is the net change in the particle's position between $t = 1$ and $t = 3$?

4. Suppose an object travels in a straight line with constant acceleration a, initial velocity v_0, and initial displacement x_0. Find a formula for the position function of the object.

5. A car brakes with constant deceleration of 5 m/s^2 produceing skid marks measuring 75 meters long before coming to a stop. How fast was the car traveling when the brakes were first applied?

6. True or false?

   (a) The antiderivative of function is unique.

   (b) If F is the antiderivative of f then f is differentiable.

   (c) If F is the antiderivative of f then $F + c$ where c is a constant is also an antiderivative.

Worksheet # 24: Review for Exam III

1. Provide a full statement of the following theorems and definitions.
   (a) The Mean Value Theorem (MVT)
   (b) Local max/min
   (c) Absolute max/min
   (d) L'Hospital's Rule
   (e) Antiderivative

2. (a) Describe in words and diagrams how to use the first derivative test to identify and classify extrema of a function $f(x)$.
   (b) Use the first derivative test to classify the extrema of the function $f(x) = 2x^3 + 3x^2 - 72x - 47$.

3. (a) Describe how to use the second derivative test. When does the test fail and what can you do if this happens?
   (b) Use the second derivative test to classify the extrema of $4x^3 + 3x^2 - 6x + 1$.

4. (a) Explain how to use the extreme value theorem (p. 272) to find the absolute maximum and absolute minimum of a continuous function $f(x)$ on a closed interval $[a, b]$.
   (b) (MA 113 Exam III, Problem 2, Spring 2009). Find the absolute minimum of the function

$$f(t) = t + \sqrt{1 - t^2}$$

   on the interval $[-1, 1]$. Be sure to specify the value of t where the minimum is attained.

5. Evaluate the following limits.
   (a) $\displaystyle\lim_{x\to-\infty} \frac{x+2}{\sqrt{9x^2+1}}$
   (b) $\displaystyle\lim_{x\to0^+} x^2 \ln x$
   (c) $\displaystyle\lim_{x\to\infty} x^2 e^x$

6. Find the most general antiderivative for each of the following.
   (a) $f(x) = 5x^{10} + 7x^2 + x + 1$
   (b) $g(x) = 2\cos(2x + 1)$
   (c) $h(x) = \dfrac{1}{2x+1}$, where $2x + 1 > 0$

7. (MA 113 Exam III, Problem 10, Spring 2001). Suppose a rectangle has one side on the x-axis and its other two vertices above the x-axis on the curve $y = 80 - x^4$. Find the dimensions of the rectangle satisfying these conditions and of largest possible area. Be sure to explain how you know you have found the absolute extreme value.

8. If $f(2) = 30$ and $f'(x) \geq 4$ for $2 \leq x \leq 6$, how small can $f(6)$ be?

9. Let $f(x) = (x - 1) + \dfrac{1}{x-1}$.
   (a) Find the y-intercept(s) of the graph of f.
   (b) Find all vertical asymptotes to the graph of f.
   (c) Compute $f'(x)$ and give the domain of $f'(x)$.
   (d) Use the first derivative to determine the intervals of increase and decrease for f and find all local maxima and local minima for f.
   (e) Compute $f''(x)$ and give the domain of $f''(x)$.

(f) Use the second derivative to find intervals of concavity for f.

(g) Sketch the graph of f and label all local extrema. Sketch the vertical asymptote(s) with dashed lines.

10. Identify each of the following as true or false.

(a) A point in the domain of f where $f'(x)$ does not exist is a critical point.

(b) Every continuous function on a closed interval will have an absolute minimum and an absolute maximum.

(c) If $f'(c) = 0$, f will have either a maximum or a minimum at c.

(d) An inflection point is an ordered pair.

(e) If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum.

(f) If $f''(c) = 0$ in the second derivative test, we must use some other method to determine if c is a min or max.

(g) A continuous function on $[a, b]$ will always have a local maximum or minimum at its endpoints.

Worksheet # 25: Area and Distance

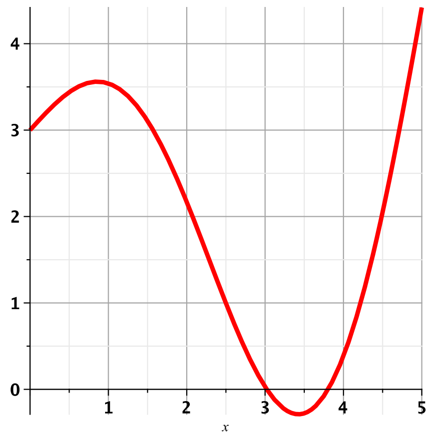
1. Write each of following in summation notation:
   (a) $1+2+3+4+5+6+7+8+9+10$
   (b) $2+4+6+8+10+12+14$
   (c) $2+4+8+16+32+64+128$.

2. Compute $\sum_{i=1}^{4}\left(\sum_{j=1}^{3}(i+j)\right)$.

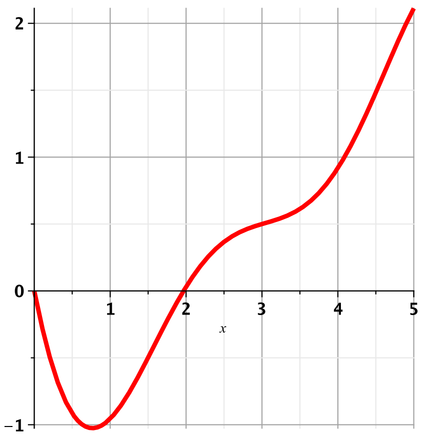
3. Find the number n such that $\sum_{i=1}^{n} i = 78$.

4. A particle starts from rest at a point P and travels with constant acceleration of 5 m/s^2 to another point Q. If it takes the particle 30 seconds to travel from P to Q what is the distance between P and Q?

5. Below is the graph of the velocity function for a particle traveling along a straight line. Use several rectangles to estimate (a) the net displacement and (b) the total distance traveled by the particle from $t=0$ to $t=5$.



6. Below is the graph of the velocity function for a particle traveling along a straight line. Use several rectangles to estimate (a) the net displacement and (b) the total distance traveled by the particle from $t=0$ to $t=5$.



7. Let A be the area under the curve $y = x^2$ from $x = 0$ to $x = 4$.

   (a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

   (b) Estimate the area by taking sample points to be midpoints and using four subintervals.

Worksheet # 26: Definite Integrals

1. (MA 113 Exam IV, Problem 5, Spring 2009). Consider the function $\dfrac{x}{x-1}$.
   (a) Compute the Riemann sum for f on the interval $[2,6]$ with $n=4$ subintervals and the left endpoints as sample points. Give the answer as a rational number (fraction).
   (b) Show that f is decreasing on $[2,6]$.
   (c) Without computing $\int_2^6 f(x)\,dx$, decide whether the Riemann sum in (a) is greater than or less than the integral.
2. Evaluate the following integrals using geometry.
   (a) $\int_0^3 \left(\frac{1}{2}x-1\right)\,dx$
   (b) $\int_{-2}^2 \sqrt{4-x^2}\,dx$
   (c) $\int_0^{10} |x-5|\,dx$
3. Suppose $\int_0^1 f(x)dx = 2$, $\int_1^2 f(x)dx = 3$, $\int_0^1 g(x)dx = -1$, and $\int_0^2 g(x)dx = 4$. Compute the following using the properties of the integral.
   (a) $\int_1^2 g(x)dx$
   (b) $\int_0^2 [2f(x)-3g(x)]dx$
   (c) $\int_1^1 g(x)dx$
   (d) $\int_1^2 f(x)dx + \int_2^0 g(x)dx$
   (e) $\int_0^2 f(x)dx + \int_2^1 g(x)dx$
4. Write the following limits of Riemann sums as definite integrals.
   (a) $\lim_{n\to\infty} \sum_{i=1}^n \frac{i^3}{n^4}$
   (b) $\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\sqrt{3+\frac{i}{n}}$
   (c) $\lim_{n\to\infty} \sum_{i=1}^n \frac{2}{n}\left(2+\frac{2i}{n}\right)^2$
5. Find $\int_0^5 f(x)dx$ if

$$f(x) = \begin{cases} 3 & x < 3 \\ x & x \geq 3 \end{cases}$$

Worksheet # 27: The Fundamental Theorem of Calculus

1. (MA 113 Exam IV, Problem 9, Spring 2008).

   (a) State both parts of the fundamental theorem of calculus. Use complete sentences.

   (b) Consider the function f on $[1, \infty)$ defined by $f(x) = \displaystyle\int_1^x \sqrt{t^5 - 1}dt$. Argue that f is increasing.

   (c) Find the derivative of the function $g(x) = \int_1^{x^3} \sqrt{t^5 - 1}dt$ on $(1, \infty)$.

2. Use Part I of the fundamental theorem of calculus to find the derivative of the following functions.

   (a) $g(x) = \displaystyle\int_1^x \left(2 + t^4\right)^5 \, dt$

   (b) $F(x) = \displaystyle\int_x^4 \cos\left(t^5\right) \, dt$

   (c) $h(x) = \displaystyle\int_0^{x^2} \sqrt[3]{1 + r^3} \, dr$

   (d) $y(x) = \displaystyle\int_{1/x^2}^0 \sin^3 t \, dt$

   (e) $G(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \sqrt{t} \sin t \, dt$

3. Use Part II of the fundamental theorem of calculus to evaluate the following integrals or explain why the theorem does not apply.

   (a) $\displaystyle\int_{-2}^5 6x \, dx$

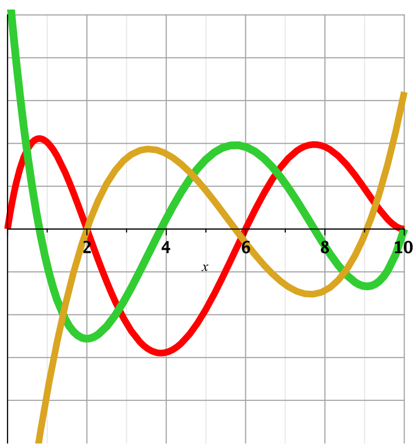
   (b) $\displaystyle\int_{-2}^7 \frac{1}{x^5} \, dx$

   (c) $\displaystyle\int_{-1}^1 e^{u+1} du$

   (d) $\displaystyle\int_0^{\pi/4} \sec^2 t \, dt$

   (e) $\displaystyle\int_{\pi/3}^{\pi/6} \frac{\sin 2x}{\sin x} \, dx$

4. Below is pictured the graph of the function $f(x)$, its derivative $f'(x)$, and an antiderivative $\int f(x) \, dx$. Identify f, f' and $\int f \, dx$.



5. Evaluate the following limits by first recognizing the sum as a Riemann sum.

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{3+\frac{i}{n}}}{n}$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} 2\frac{(2+\frac{2i}{n})^2}{n}$

Worksheet # 28: Indefinite Integrals and the Net Change Theorem

1. Compute the definite integral.

   (a) $\displaystyle\int_0^2 4x^5 + x^2 + 2x + 1 \; dx$

   (b) $\displaystyle\int_0^{\pi/2} (\sin x + 5\cos x) \; dx$

   (c) $\displaystyle\int_1^{16} \frac{1+\sqrt{x}}{\sqrt{x}} \; dx$

   (d) $\displaystyle\int_1^2 \sqrt{\frac{7}{x^3}} \; dx$

2. Find the general indefinite integral.

   (a) $\displaystyle\int \frac{15}{x} dx$

   (b) $\displaystyle\int \frac{x^2 - \sqrt{x}}{x} \; dx$

   (c) $\displaystyle\int \cos(x) - \sin(x) + e^x \; dx$

   (d) $\displaystyle\int (1 + \tan^2 \theta) \; d\theta$

   (e) $\displaystyle\int \sin^2 y \; dy$ [Hint: Use an identity.]

3. Let the velocity of a particle traveling along the x-axis be given by $v(t) = t^2 - 3t + 8$. Find the displacement and distance traveled by the particle from $t = 2$ to $t = 4$ seconds.

4. The velocity of a particle traveling along the x-axis is given by $v(t) = 3t^2 + 8t + 15$ and the particle is initially located 5 m left of the origin. How far does the particle travel from $t = 2$ seconds to $t = 3$ seconds? After 3 seconds where is the particle with respect to the origin?

5. (MA 113 Exam IV, Problem 7, Spring 2009). A particle is traveling along a straight line so that its velocity at time t is given by $v(t) = 4t - t^2$ (measure in meters per second).

   (a) Graph the function $v(t)$.

   (b) Find the total distance traveled by the particle during the time period $0 \le t \le 5$.

   (c) Find the net distance traveled by the particle during the time period $0 \le t \le 5$.

6. An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

7. (Similar to problem 47, p. 397). Draw the region R that lies between the y-axis and the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$. To find the area between a continuous function f and the x-axis on the interval $[a, b]$, we just evaluate $\displaystyle\int_a^b f(x) \; dx$. Use some intuition to find the area of R.

Worksheet # 29: The Substitution Rule

1. Evaluate the following indefinite integrals. Be sure to indicate any substitutions that you use.

   (a) $\displaystyle\int \frac{4}{(1+2x)^3}\, dx$

   (b) $\displaystyle\int x^2\sqrt{x^3+1}\, dx$

   (c) $\displaystyle\int \cos^4\theta \sin\theta\, d\theta$

   (d) $\displaystyle\int \frac{1}{(5t+4)^{2.7}}\, dt$

   (e) $\displaystyle\int \sec 2\theta \tan 2\theta\, d\theta$

   (f) $\displaystyle\int \sec^3 x \tan x\, dx$

   (g) $\displaystyle\int x^2 \cdot \sqrt[3]{x^3+1}\, dx$

   (h) $\displaystyle\int \frac{dx}{2x+1}$

2. Evaluate the following definite integrals. Be sure to indicate any substitutions that you use.

   (a) $\displaystyle\int_0^7 \sqrt{4+3x}\, dx$

   (b) $\displaystyle\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\, dt$

   (c) $\displaystyle\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$

   (d) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x^2}}\, dx$

   (e) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

   (f) $\displaystyle\int_0^{\pi} x\cos x^2 dx$

   (g) $\displaystyle\int_0^3 e^x \sin(e^x) dx$

   (h) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

3. If f is continuous and $\int_0^9 f(x)\, dx = 4$ find $\int_0^3 x \cdot f(x^2)\, dx$.

4. If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x)\, dx$.

5. Identify each of the following statements as true or false. Justify your answer.

   (a) If f and g are continuous on $[a,b]$, then $\int_a^b [f(x)g(x)]dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right)$.

   (b) If f is continuous on $[a,b]$, then $\int_a^b 5f(x)\, dx = 5\int_a^b f(x)\, dx$.

(c) If f' is continuous on $[1,3]$, then $\int\limits_1^3 f'(v)\ dv = f(3) - f(1)$.

(d) $\displaystyle\int_{-1}^{1} \left(x^5 - 6x^9 + \frac{\sin x}{(1+x^4)^2} \right) dx = 0$.

(e) $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\ dx = 2\int_{0}^{5} (ax^2 + c)\ dx$.

(f) If f is continuous on $[a,b]$, then $\dfrac{d}{dx}\left(\int\limits_a^b f(x)\ dx \right) = f(x)$.

Worksheet # 30: Review for Exam IV

1. Compute the derivative of the given function.

   (a) $f(\theta) = \cos(2\theta^2 + \theta + 2)$

   (b) $g(u) = \ln(\sin^2 u)$

   (c) $h(x) = \displaystyle\int_{-3599}^{x} t^2 - te^{t^2+t+1}\,dt$

   (d) $r(y) = \arccos(y^3 + 1)$

2. Compute the following definite integrals.

   (a) $\displaystyle\int_{-1}^{1} e^{u+1} du$

   (b) $\displaystyle\int_{-2}^{2} -\sqrt{4 - x^2} dx$

   (c) $\displaystyle\int_{1}^{9} \frac{x-1}{\sqrt{x}} dx$

   (d) $\displaystyle\int_{0}^{10} |x - 5| dx$

   (e) $\displaystyle\int_{0}^{\pi} \sec^2(t/4) dt$

   (f) $\displaystyle\int_{0}^{1} xe^{-x^2} dx$

3. Provide the most general antiderivative of the following functions.

   (a) $x^4 + x^2 + x + 1000$

   (b) $(3x - 2)^{20}$

   (c) $\dfrac{\sin(\ln(x))}{x}$

4. Use implicit differentiation to find $\frac{dy}{dx}$.

   (a) $x^2 + xy + y^2 = 16$

   (b) $x^2 + 2xy - y^2 + x = 2$. Also, compute $\frac{dy}{dx}(1, 2)$

5. If $F(x) = \displaystyle\int_{3x^2+1}^{7} \cos t\, dt$ find $F'(x)$. Justify your work carefully.

6. Suppose a bacteria colony grows at a rate of $r(t) = 100 \ln(2) 2^t$ with t in hours. By how many bacteria does the population increase from time $t = 1$ to $t = 3$?

7. Use a left Riemann sum with 4 equal subintervals to estimate the value of $\displaystyle\int_{1}^{5} x^2 dx$. Will this estimate be larger or smaller than the actual value of definite integral? Explain.

8. A conical tank with radius 5 m and height 10 m is being filled with water at a rate of 3 m^3 per minute. How fast is the water level increasing when the height is 3?

9. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of the container is twice its width. Material for the base costs \$ 10 per square meter while material for the sides costs \$ 6 per square meter. Find the materials cost for the cheapest possible container.

10. State the mean value theorem. Then if $3 \leq f'(x) \leq 5$ for all x, find the maximum possible value for $f(8) - f(2)$.