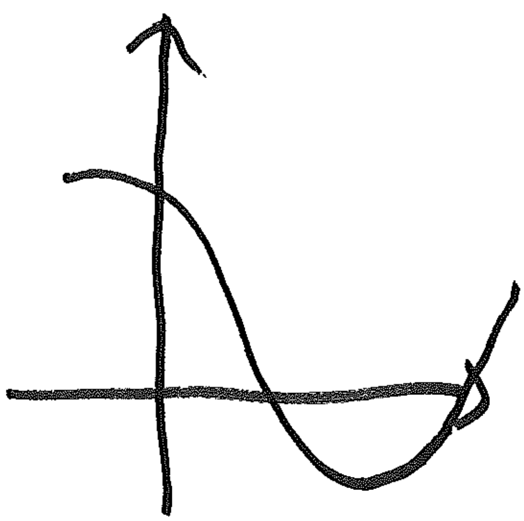
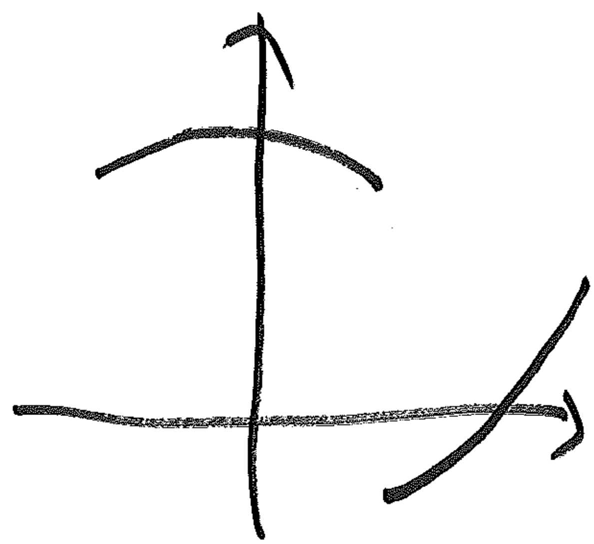
CONTINUITY

INFORMALLY, THE GRAPH OF THE FUNCTION CAN BE DRAWN WITHOUT PICKING UP THE PENCIL





RIGUROUSLY: f IS CONTINUOUS AT a IF $\lim_{x \to a} f(x) = f(a)$

THUS: f MUST BE DEFINED NEAR a, INCLUDING a

$\lim_{x \to a} f(x)$ MUST EXIST

$\lim_{x \to a} f(x) = f(a)$

EX: $f(x) = x^2 + 2x + 5$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 2x + 5) = \lim_{x \to 1} x^2 + \lim_{x \to 1} 2x + \lim_{x \to 1} 5 = 1^2 + 2 \cdot 1 + 5 = f(1)$$

BAD GUYS

1) $f(x) = \begin{cases} x+1, & x < 1 \\ C, & x = 1 \\ x^2+1, & x > 1 \end{cases}$

FOR WHAT VALUES OF C IS f CONTINUOUS AT 1?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1 + 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2+1) = 1^2 + 1 = 2$$

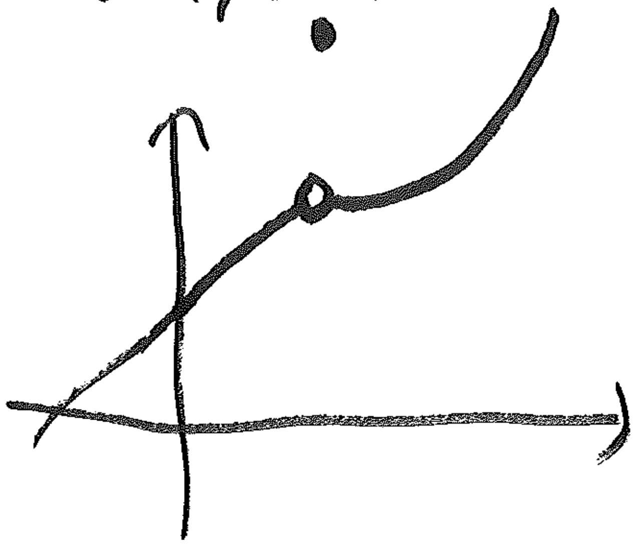
$$\lim_{x \to 1} f(x) = 2 \stackrel{?}{=} f(1)$$

NEED: $C = 2$

$$f(x) = \begin{cases} x+1, & x<1 \\ 5, & x=1 \\ x^2+1, & x>1 \end{cases}$$

REMOVABLE DISCONTINUITY

$$\lim_{x \to a} f(x) \neq f(a)$$



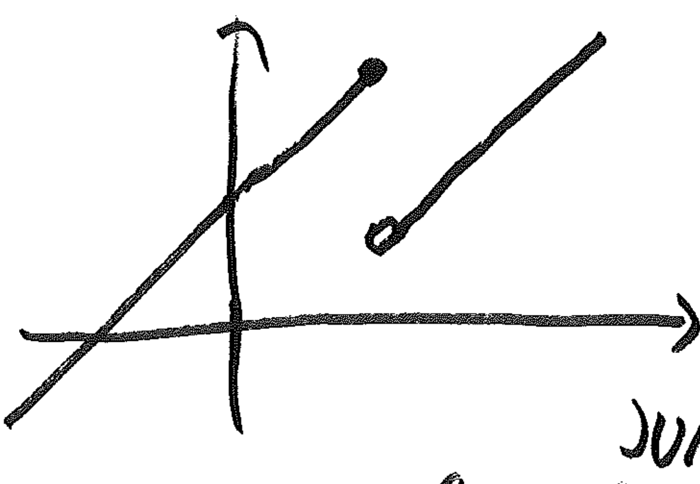
CAN REDEFINE $f(x) = \begin{cases} x+1, & x<1 \\ 2, & x=1 \\ x^2+1, & x>1 \end{cases}$

CONTINUOUS AT 1

EX: $f(x) = \begin{cases} x+1, & x \leq 1 \\ x, & x>1 \end{cases}$

$$\lim_{x \to 1^-} f(x) = 2$$

$$\lim_{x \to 1^+} f(x) = 1$$



JUMP DISCONTINUITY

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

EX: $f(x) = \begin{cases} \frac{1}{x-1}, & x > 1 \\ 0, & x \le 1 \end{cases}$

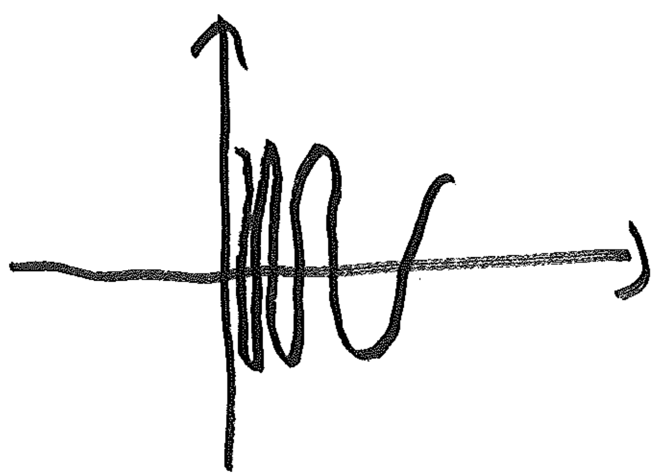
$\lim_{x \to 1^+} f(x) = +\infty$ INFINITE DISCONTINUITY

$\lim_{x \to a^+} f(x) = \pm\infty$

OR

$\lim_{x \to a^-} f(x) = \pm\infty$

EX: $f(x) = \begin{cases} \sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$



TYPE II DISCONTINUITY

$\lim_{x \to a^+} f(x)$ DNE

OR

$\lim_{x \to a^-} f(x)$ DNE

DEF: f CONTINUOUS ON AN INTERVAL IF f IS CONTINUOUS AT EVERY POINT IN THE INTERVAL

AT THE ENDPOINTS, ONLY LOOK AT ONE-SIDED LIMITS

EX: $f : [2,3] \to \mathbb{R}$

$$f(x) = \sqrt{x-2}$$

FOR $2 < a < 3$: $\lim\limits_{x \to a} \sqrt{x-2} \stackrel{?}{=} \sqrt{a-2}$ ✓

FOR $a=2$: $\lim\limits_{x \to 2^+} \sqrt{x-2} \stackrel{?}{=} \sqrt{2-2} = 0$ ✓

FOR $a=3$: $\lim\limits_{x \to 3^+} \sqrt{x-2} = \sqrt{3-2} = 1$ ✓

$f : (0,1) \to \mathbb{R}$

$f(x) = \frac{1}{x}$ CONTINUOUS

$f : [0,1] \to \mathbb{R}$ $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ C, & x = 0 \end{cases}$ NOT CONTINUOUS

THM: LET f, g BE TWO CONTINUOUS FUNCTIONS ON THE SAME INTERVAL, c SOME CONSTANT. THEN:

$f+g$, $f-g$, cf, fg ARE CONTINUOUS

$\frac{f}{g}$ IS CONTINUOUS AT ALL POINTS x_0 WHERE $g(x_0) \neq 0$

$\sqrt[n]{f}$ IS CONTINUOUS IF WELL-DEFINED

IN PARTICULAR, POLYNOMIALS ARE ~~WELL-DEFIN~~ CONTINUOUS ON $(-\infty, +\infty)$, RATIONAL FNS. $\frac{P(x)}{Q(x)}$, n^{TH} ROOTS ARE CONTINUOUS ON THEIR DOMAINS

THM: IF $\lim_{x \to a} g(x) = g(a)$, AND $\lim_{x \to g(a)} f(x) = f(g(a))$. THEN $\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)$

INVERSES OF CONTINUOUS FUNCTIONS (IF DEFINED) ARE ALSO CONTINUOUS

EX: $f(x) = \sqrt[3]{x^2+4}$ CONTINUOUS EVERYWHERE

$f(x) = \sqrt{x^2-1}$ DOMAIN: $(-\infty,-1] \cup [1,+\infty)$

CONTINUOUS ON DOMAIN

EXPONENTIAL FNS ARE CONTINUOUS ON $(-\infty,+\infty)$

TRIG. FNS ARE CONTINUOUS ON THEIR DOMAIN

$\log_b$ ARE CONTINUOUS ON $(0,+\infty)$

INVERSE TRIG FN CONTINUOUS ON THEIR DOMAIN

EX: $f(x) = \ln(4-\sqrt{x})$ $\qquad 4-\sqrt{x}>0$

DOMAIN: $0 \le x < 16$, CONTINUOUS ON $[0,16)$

$$\lim_{x\to 2} f(x) = \ln(4-\sqrt{2})$$

IVT $\quad f:[a,b]\to\mathbb{R}$ CONTINUOUS

PICK N BETWEEN $f(a)$ AND $f(b)$

THEN THERE IS SOME $a \le c \le b$

SO THAT $f(c) = N$