THE DERIVATIVE AS A FUNCTION

RECALL : $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

CAN VARY a

CAN DEFINE f' AS A FUNCTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DEF: f IS DIFFERENTIABLE ON (a, b)
IF $f'(x)$ EXISTS FOR ALL $a < x < b$

REMARK: THE DOMAIN OF f' IS AT MOST
THE DOMAIN OF f, BUT COULD BE SMALLER

EX: $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

DOMAIN OF f : $(-\infty, +\infty)$

DOMAIN OF f' : $(-\infty, 0) \cup (0, +\infty)$

<u>EX</u>: $f(x) = 2x+5$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)+5-(2x+5)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x}+2h+\cancel{5}-\cancel{2x}-\cancel{5}}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = 2$$

<u>EX</u>: $f(x) = \sqrt{x+2}$ DOMAIN : $[-2, +\infty)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h+2} - \sqrt{x+2})(\sqrt{x+h+2} + \sqrt{x+2})}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{x+h+2-(x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+2} + \sqrt{x+2}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}}$$

$$f'(x) = \frac{1}{2\sqrt{x+2}} \quad \text{DOMAIN: } (-2, +\infty)$$

<u>EX</u>: $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$\boxed{f'(x) = -\frac{1}{x^2}}$$

LEIBNIZ NOTATION

$$y = f(x) \qquad f' = \frac{df}{dx} = \frac{dy}{dx}$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

HIGHER ORDER DERIVATIVES

$$f(x),\ f'(x),\ f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} \quad \ldots$$

$$\frac{d^2y}{dx^2} = f'' \qquad \frac{d^3y}{dx^3} = f''' \quad \ldots$$

<u>EX</u>: $f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{(x+h)^2} - \left(-\frac{1}{x^2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h(x+h)^2 x^2}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h(x+h)^2 x^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}(x+h)^2 x^2} = \lim_{h \to 0} \frac{2x+h}{(x+h)^2 x^2}$$

$$= \frac{2x}{x^2 \cdot x^2} = \frac{2}{x^3}$$

$$f''(x) = \frac{2}{x^3}$$

WHEN DOES $f'(a)$ NOT EXIST?

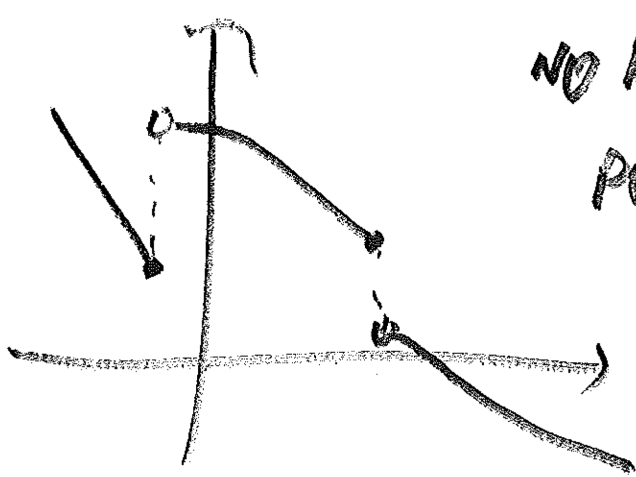
THM: IF $f'(a)$ EXISTS, THEN f MUST BE CONTINUOUS AT a

THUS: IF f IS NOT CONTINUOUS AT a, $f'(a)$ DNE

PROOF: $\lim\limits_{x \to a} (f(x) - f(a)) = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot (x-a)$

$$= \underbrace{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}_{f'(a)} \cdot \underbrace{\lim_{x \to a} (x-a)}_{0} = f'(a) \cdot 0 = 0$$

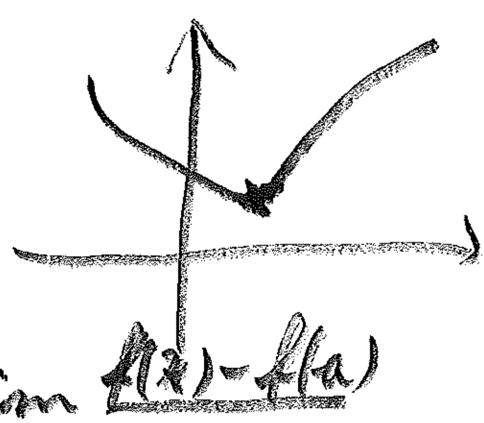
$$\lim_{x \to a} f(x) = f(a)$$

EX: 

NO DERIVATIVE AT POINTS OF DISCONTINUITY

$f(x) = |x| \qquad f'(0)$ DNE

CORNER/KINK

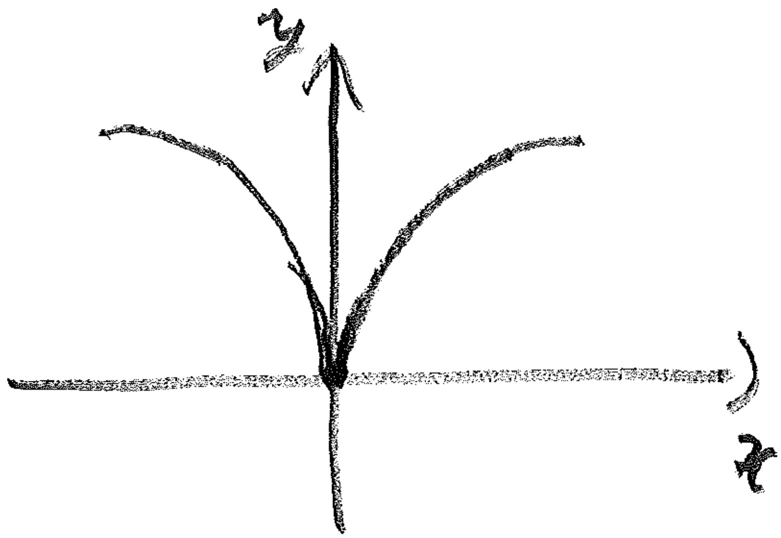


$$\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} \neq \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$

EX: $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = \pm\infty$

OR

$$\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \pm\infty$$



$f(x) = x^{\frac{2}{3}}$

TANGENT LINE IS VERTICAL

$$\lim_{x \to 0^+} \frac{x^{\frac{2}{3}} - 0^{\frac{2}{3}}}{x} = \lim_{x \to 0^+} \frac{1}{x^{\frac{1}{3}}} = +\infty$$