PRODUCT/QUOTIENT RULE

PICK f, g DIFFERENTIABLE AT x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

THUS f, g CONTINUOUS AT x

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h) \cdot \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x) f'(x) + f(x) g'(x)$$

THUS: $(fg)' = f'g + fg'$

Ex: $(x^2)' = (x \cdot x)' = x' \cdot x + x \cdot x' = 1 \cdot x + x \cdot 1 = 2x$

$(x^3 e^x)' = (x^3)'(e^x) + x^3 (e^x)' = 3x^2 e^x + x^3 e^x$
$= (3x^2 + x^3) e^x$

$(x^3 e^x)'' = ((3x^2 + x^3) e^x)' = (3x^2 + x^3)' e^x +$
$+ (3x^2 + x^3)(e^x)' = (6x + 3x^2) e^x$
$+ (3x^2 + x^3) e^x = (6x + 6x^2 + x^3) e^x$

Ex: $(e^{2x})' = (e^x \cdot e^x)' = (e^x)' \cdot e^x + e^x \cdot (e^x)'$
$= e^{2x} + e^{2x} = 2e^{2x}$

$(fgh)' = f'(gh) + f(gh)' = f'gh + f(g'h + gh')$
$= f'gh + fg'h + fgh'$

$$(x^n)' = (\underbrace{x \cdot \ldots \cdot x}_{n \text{ TIMES}})' = x' \cdot x \cdot \ldots \cdot x + x \cdot x' \cdot x \cdot \ldots \cdot x + \ldots + x \cdot x \cdot \ldots \cdot x'$$
$$= \underbrace{x^{n-1} + x^{n-1} + \ldots + x^{n-1}}_{n \text{ TIMES}} = n x^{n-1}$$

QUOTIENT RULE

$$\left(\frac{f}{g}\right)' \stackrel{?}{=} \frac{f'}{g'} \quad \times \quad \text{NO!}$$

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x+h)\,g(x)}$$

$$\lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x)}{h\,g(x+h)\,g(x)} = \lim_{h\to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h\,g(x+h)} = \lim_{h\to 0} \frac{1}{g(x+h)} \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{1}{g(x)} \cdot f'(x)$$

$$\lim_{h\to 0} \frac{f(x)g(x)-f(x)g(x+h)}{h\,g(x+h)g(x)} = \lim_{h\to 0} \frac{f(x)\cdot[g(x)-g(x+h)]}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{f(x)}{g(x+h)g(x)} \cdot \lim_{h\to 0} \frac{g(x)-g(x+h)}{h}$$

$$= \frac{f(x)}{g^2(x)} \cdot (-g'(x))$$

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}}$$

PARTICULAR CASE:

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

<u>EX</u>: $(x^{-n})' \xlongequal{\text{POWER RULE}} (-n)\cdot x^{-n-1}$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{-(x^n)'}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}}$$

$$= \frac{-n}{x^{n+1}} = (-n)\cdot x^{-n-1}$$

CAN DIFFERENTIATE ANY RATIONAL FUNCTION

EX: $f(x) = \dfrac{\sqrt{x} - 2x^3}{x^2+1}$

$$f'(x) = \frac{(\sqrt{x} - 2x^3)'(x^2+1) - (\sqrt{x} - 2x^3)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(\frac{1}{2}x^{-\frac{1}{2}} - 6x^2)(x^2+1) - (x^{\frac{1}{2}} - 2x^3)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{\frac{1}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} - 6x^4 - 6x^2 - 2x^{\frac{3}{2}} + 4x^4}{(x^2+1)^2}$$

EX: LET $f(x) = \dfrac{e^x}{1+x}$. FIND ALL POINTS WHERE THE TANGENT TO THE GRAPH IS HORIZONTAL

FIND ALL x SO THAT $f'(x) = 0$

$$f'(x) = \left(\frac{e^x}{1+x}\right)' = \frac{(e^x)'\cdot(1+x) - e^x\cdot(1+x)'}{(1+x)^2}$$

$$= \frac{e^x\cdot(1+x) - e^x}{(1+x)^2} = \frac{\cancel{e^x} + xe^x - \cancel{e^x}}{(1+x)^2} = \frac{xe^x}{(1+x)^2}$$

SOLVE: $\frac{xe^x}{(1+x)^2} = 0$

$$xe^x = 0 \mid \cdot \frac{1}{e^x} \qquad x = 0$$

<u>REMARK</u>: SOMETIMES IT IS EASIER TO SIMPLIFY FIRST

<u>Ex</u>: $f(x) = \frac{x^{\frac{1}{3}} + 3xe^x}{x}$

QUOTIENT RULE WORKS, BUT LONG

$$f(x) = \frac{x^{\frac{1}{3}}}{x} + \frac{3xe^x}{x} = x^{-\frac{2}{3}} + 3e^x$$