DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$f(x)=\sin x$, $f'(x)=?$

$$f'(0)=\lim_{h\to 0}\frac{\sin h-\sin 0}{h}=\lim_{h\to 0}\frac{\sin h}{h}$$

FOR $0<h<\frac{\pi}{2}$, WE HAVE

$$h\cos h<\sin h<h$$

$$\cos h<\frac{\sin h}{h}<1$$

$$\lim_{h\to 0}\cos h=\cos 0=1$$

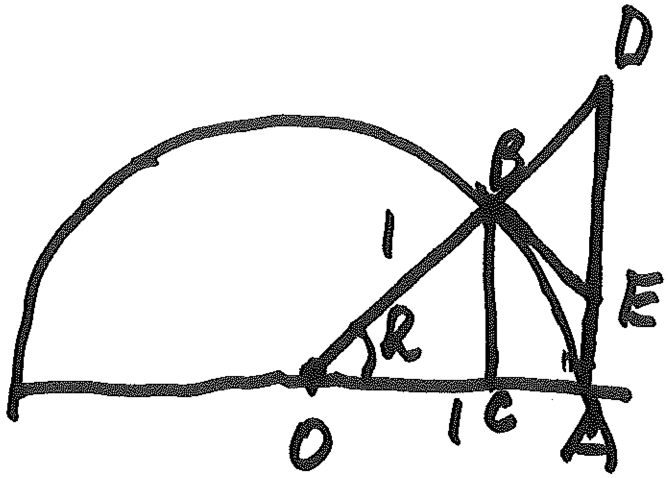
$$\lim_{h\to 0}1=1$$

BY SQUEEZE THM, $\lim_{h\to 0^+}\frac{\sin h}{h}=1$

$$\frac{\sin(-h)}{-h}=\frac{-\sin h}{-h}=\frac{\sin h}{h}$$ EVEN FUNCTION

$$\lim_{h\to 0^-}\frac{\sin h}{h}=\lim_{h\to 0^+}\frac{\sin h}{h}=1$$

GEOMETRICALLY



h = LENGTH OF ARC AB

$\sin h$ = LENGTH OF BC

$|BC| < \text{ARC}(AB) \qquad \sin h < h$

$h < \tan h$

$$\tan h = \frac{|AD|}{|OA|} = |AD|$$

$$\text{ARC}(AB) < |BE| + |EA| < |DE| + |EA| = |AD|$$

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

EX: $\lim_{x \to 0} \frac{\sin(9x)}{4x} = \lim_{x \to 0} \frac{\sin(9x)}{9x} \cdot \frac{9x}{4x}$

$= \lim_{x \to 0} \frac{\sin(9x)}{9x} \cdot \lim_{x \to 0} \frac{9}{4} = 1 \cdot \frac{9}{4} = \frac{9}{4}$

$9x = h$, $\lim_{x \to 0} \frac{\sin(9x)}{9x} = \lim_{h \to 0} \frac{\sin h}{h} = 1$

IN GENERAL, $\lim_{x \to 0} \frac{\sin(f(x))}{g(x)}$

IF $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$, THEN

$\lim_{x \to 0} \frac{\sin(f(x))}{g(x)} = \lim_{x \to 0} \frac{\sin(f(x))}{f(x)} \cdot \frac{f(x)}{g(x)}$

$= \lim_{x \to 0} \frac{\sin(f(x))}{f(x)} \cdot \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)}$

WHAT ABOUT $\cos'(0)$?

$$\cos'(0) = \lim_{h \to 0} \frac{\cos h - \cos 0}{h} = \lim_{h \to 0} \frac{\cos h - 1}{h}$$

$$= \lim_{h \to 0} \frac{-2 \sin^2 \frac{h}{2}}{h} = \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\left(\frac{h}{2}\right)^2} \cdot \frac{\left(\frac{h}{2}\right)^2 \cdot (-2)}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \cdot \lim_{h \to 0} \frac{-h^2}{2h} = 1 \cdot 0 = 0$$

LET $f(x) = \sin x$; WE COMPUTED $f'(0) = 1$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cdot (\cos h - 1) + \cos x \cdot \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h \to 0} \frac{\cos x \cdot \sin h}{h}$$

$$= (\sin x) \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + (\cos x) \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

$(\cos x)' = -\sin x$ SIMILARLY

Ex: $\dfrac{d}{dx}(\sin^2 x) = \dfrac{d}{dx}(\sin x \cdot \sin x)$

$$= \left(\frac{d}{dx}\sin x\right)\cdot \sin x + (\sin x)\cdot \frac{d}{dx}\sin x$$

$$= (\cos x)\cdot \sin x + \sin x \cdot \cos x = 2\sin x \cos x = \sin(2x)$$

$$\frac{d}{dx}(\cos^2 x) = \frac{d}{dx}(1-\sin^2 x) = \frac{d}{dx}1 - \frac{d}{dx}(\sin^2 x)$$

$$= 0 - \sin(2x) = -\sin(2x)$$

Ex: $\dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right)$

$$= \frac{\frac{d}{dx}(\sin x)\cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad , \quad x \neq \frac{\pi}{2} + k\pi$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{-\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x$$

<u>EX</u>: WHAT ABOUT $\sin(x^2)$?

$$\frac{d}{dx}\left[\sin(x^2)\right] = \frac{d}{dx}(f \circ g)$$

VS.

$(\sin x)^2$ = PRODUCT OF TWO FUNCTIONS

$g(x) = x^2$, $f(x) = \sin x$