CHAIN RULE

RECALL: $(f \circ g)(x) = \underset{\text{OUTER}}{f}(\underset{\text{INNER}}{g(x)})$

$f \circ g \neq g \circ f$

EX: $f(x) = e^x$, $g(x) = x^3 + 1$

$$(f \circ g)(x) = e^{g(x)} = e^{x^3+1} = e\, e^{x^3}$$

$$(g \circ f)(x) = f(x)^3 + 1 = (e^x)^3 + 1 = e^{3x} + 1$$

GOAL: COMPUTE $(f \circ g)'(x)$ ASSUME f, g DIFFERENTIABLE

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{(f \circ g)(x+h) - (f \circ g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = f'(g(x))$$

$$g(x+h) - g(x) = u$$

$$\lim_{h \to 0} [g(x+h) - g(x)] = \lim_{h \to 0} g(x+h) - \lim_{h \to 0} g(x)$$

$$= g(x) - g(x) = 0$$

$$g(x+h) = g(x) + u$$

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = \lim_{u \to 0} \frac{f(g(x)+u) - f(g(x))}{u}$$

$$= f'(g(x))$$

THUS

$$\boxed{(f \circ g)'(x) = f'(g(x))\, g'(x)}$$

ALMOST WORKS; PROBLEM IF $g(x+h) - g(x) = 0$

ALGORITHMICALLY: $h(x) = f \circ g$

1) IDENTIFY f AND g CORRECTLY

2) COMPUTE f'

3) EVALUATE f' AT $g(x)$

4) COMPUTE g'

5) COMPUTE $h'(x) = f'(g(x)) \cdot g'(x)$

LEIBNIZ NOTATION

$u = g(x)$, $y = f(g(x)) = f(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

EX: $h(x) = (x+1)^{1000}$

1) $h = f(g(x))$ $\quad g(x) = x+1$

$f(x) = x^{1000}$

2) $f'(x) = 1000 \cdot x^{999}$

3) $f'(x+1) = 1000\,(x+1)^{999}$

4) $g'(x) = (x+1)' = 1$

5) $h'(x) = 1000(x+1)^{999} \cdot 1 = 1000(x+1)^{999}$

IN GENERAL, LET $u = g(x)$

$$\frac{d}{dx}(u^n) = n u^{n-1} \cdot \frac{du}{dx}$$

Ex: $(\sin^4 x)' = 4\sin^3 x \cdot (\sin x)' = 4\sin^3 x \cdot \cos x$

Ex: $[\sin(x^2)]'$

1) $f(x) = \sin x, \quad g(x) = x^2$

2) $f'(x) = \cos x$

3) $f'(g(x)) = \cos(g(x)) = \cos(x^2)$

4) $g'(x) = 2x$

5) $[\sin(x^2)]' = [\cos(x^2)] \cdot 2x = 2x\cos(x^2)$

IN GENERAL, $u = g(x)$

$$(\sin u)' = (\cos u) \cdot u'$$

<u>EX</u>: $$[\sin(\tan x)]' = \cos(\tan x) \cdot (\tan x)'$$

$$= \cos(\tan x) \cdot \sec^2 x$$

<u>EX</u>: $$h(x) = \sqrt{1 + \sqrt[3]{3 + \cos^2 x}}$$

1) $h(x) = f(g(x))$, $f(x) = \sqrt{x}$, $g(x) = 1 + \sqrt[3]{3 + \cos^2 x}$

2) $f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2} x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

3) $f'(g(x)) = \dfrac{1}{2\sqrt{g(x)}} = \dfrac{1}{2\sqrt{1 + \sqrt[3]{3 + \cos^2 x}}}$

4) $g'(x) = \left(\sqrt[3]{3 + \cos^2 x}\right)' = \frac{1}{3}(3 + \cos^2 x)^{-\frac{2}{3}} \cdot (-\sin(2x))$

$g(x) = z(w(x))$, $z(x) = \sqrt[3]{x}$

$w(x) = 3 + \cos^2 x$

$z'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$z'(w(x)) = \frac{1}{3}(3 + \cos^2 x)^{-\frac{2}{3}}$

$$w'(x) = (3+\cos^2 x)' = (\cos^2 x)'$$

$$= (\cos x \cdot \cos x)' = (\cos x)' \cos x + (\cos x)(\cos x)'$$

$$= -2 \sin x \cos x = -\sin(2x)$$

5) $$h'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{1+\sqrt[3]{3+\cos^2 x}}} \cdot \frac{1}{3}(3+\cos^2 x)^{-\frac{2}{3}} \cdot (-\sin 2x)$$

<u>REMARK</u>: $(f \circ g)'(5)$ DEPENDS ON WHAT?

$$(f \circ g)'(5) = f'(g(5)) \cdot g'(5)$$

NEED TO KNOW: $g(5)$ $g'(5)$ $f'(g(5))$

<u>EX</u>: $g(5) = 7$, $g'(5) = -3$, $f'(7) = 3$

$$(f \circ g)'(5) = f'(g(5)) \cdot g'(5)$$

$$= f'(7) \cdot (-3) = 3 \cdot (-3) = -9$$