IMPLICIT DIFFERENTIATION

$y = f(x)$ EXPLICIT DEPENDENCE ON x

__EX__: $y = \sqrt{1-x^2}$

SOMETIMES THE DEPENDENCE OF y ON x IS __IMPLICIT__

__EX__: $y^2 + x^2 = 1$

CAN SOMETIMES SOLVE FOR y EXPLICITLY

$y^2 = 1 - x^2$, $y = \pm\sqrt{1-x^2}$

__EX__: $y^5 - e^{x^2} y + \cos(x + y^2) = 0$

CANNOT FIND y EXPLICITLY

EXPLICIT FORM

$y = \sqrt{1-x^2}$ $\quad \frac{dy}{dx} = \frac{d}{dx}(1-x^2)^{\frac{1}{2}} = \frac{1}{2\sqrt{1-x^2}} \cdot$

$$\cdot \frac{d}{dx}(1-x^2) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$y^2 + x^2 = 1 \quad \Big| \frac{d}{dx}$$

$$\frac{d}{dx}(y^2 + x^2) = \frac{d}{dx} 1$$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) = 0$$

$$2y \frac{dy}{dx} + 2x = 0$$

$$2y \frac{dy}{dx} = -2x \quad \Big| \cdot \frac{1}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

EX: $y^5 + x^3 = 2xy$

WHAT IS $\frac{dy}{dx}$?

DON'T NEED TO SOLVE FOR y

INSTEAD, TAKE $\frac{d}{dx}$

$$\frac{d}{dx}(y^5 + x^3) = \frac{d}{dx}(2xy)$$

$$\frac{d}{dx}y^5 + \frac{d}{dx}x^3 = \frac{d}{dx}(2x) \cdot y + (2x) \cdot \frac{dy}{dx}$$

$$5y^4 \cdot \frac{dy}{dx} + 3x^2 = 2y + 2x\frac{dy}{dx}$$

$$5y^4\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 3x^2$$

$$(5y^4 - 2x)\frac{dy}{dx} = 2y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{2y - 3x^2}{5y^4 - 2x}}$$

Q: WHAT IS THE SLOPE OF THE TANGENT LINE TO $y^5+x^3=2xy$ AT $(1,1)$?

PLUG IN $x=1, y=1$ $\quad 1^5+1^3=2\cdot1\cdot1$ ✓

RECALL: $\frac{dy}{dx}=\frac{2y-3x^2}{5y^4-2x}$

WHEN $x=y=1$: $\frac{dy}{dx}=\frac{2\cdot1-3\cdot1^2}{5\cdot1^4-2\cdot1}=\frac{-1}{3}=-\frac{1}{3}$

TANGENT LINE AT $(1,1)$:

$$y-1=-\tfrac{1}{3}(x-1)$$

ALGORITHMICALLY:

1) DIFFERENTIATE WITH RESPECT TO x
2) COLLECT ALL TERMS WITH $\frac{dy}{dx}$
3) SOLVE FOR $\frac{dy}{dx}$
4) IF NECESSARY: PLUG IN VALUES FOR x, y

EX: $y^4 - x^2 + x = 1$

FIND THE TANGENT LINE AT $(0,-1)$

1) TAKE $\frac{d}{dx}$

$$\frac{d}{dx}(y^4 - x^2 + x) = \frac{d}{dx} 1$$

$$4y^3 \cdot \frac{dy}{dx} - 2x + 1 = 0$$

2) $4y^3 \frac{dy}{dx} = 2x - 1$

3) $\frac{dy}{dx} = \frac{2x-1}{4y^3}$

SLOPE AT $(0,-1) = \frac{2 \cdot 0 - 1}{4 \cdot (-1)^3} = \frac{-1}{-4} = \frac{1}{4}$

TANGENT LINE: $y - (-1) = \frac{1}{4}(x - 0)$

$y + 1 = \frac{1}{4}x$ IMPLICIT

$y = \frac{1}{4}x - 1$ EXPLICIT

EX: $xy^2 - \tan y = x^3$

1) DIFF. WRT x

$$\frac{d}{dx}(xy^2 - \tan y) = \frac{d}{dx}x^3$$

$$\frac{d}{dx}(xy^2) - \frac{d}{dx}(\tan y) = 3x^2$$

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}x \cdot y^2 + x \cdot \frac{d}{dx}y^2$$

$$= y^2 + x \cdot 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(\tan y) = \sec^2 y \cdot \frac{dy}{dx}$$

$$y^2 + 2xy\frac{dy}{dx} - \sec^2 y \cdot \frac{dy}{dx} = 3x^2$$

2) $2xy\frac{dy}{dx} - \sec^2 y \cdot \frac{dy}{dx} = 3x^2 - y^2$

3) $(2xy - \sec^2 y)\frac{dy}{dx} = 3x^2 - y^2$ $\quad \frac{dy}{dx} = \frac{3x^2 - y^2}{2xy - \sec^2 y}$

INVERSE FUNCTIONS

$y = f(x)$, EXPRESS $x = f^{-1}(y)$

1) $\frac{d}{dx} x = \frac{d}{dx} [f^{-1}(y(x)))$

$$1 = \left[\frac{d}{dx} f^{-1}\right](y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{d}{dx} f^{-1}\right)(f(x))}$$

EX:

$y = \arcsin x = f(x) \qquad f^{-1}(x) = \sin x$

$$\frac{dy}{dx} = \frac{1}{\left[\frac{d}{dx} \sin x\right](\arcsin x)} = \frac{1}{\cos(\arcsin x)}$$

$$\cos(\arcsin x) = \sqrt{1 - x^2}$$

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}}$$

Ex: $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

Ex: $\frac{d}{dx}(\arctan x)$

$f(x) = \arctan x$, $f^{-1}(x) = \tan x$

$$\frac{d}{dx}(\arctan x) = \frac{1}{\left(\frac{d}{dx}\tan\right)(\arctan x)}$$

$$= \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)}$$

$\cancel{1+\sec^2 y = \tan^2}$ $1+\tan^2 y = \sec^2 y$

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$$