DERIVATIVE OF LOGARITHM

RECALL: $y = f(x)$

$$x = f^{-1}(y) \qquad \frac{dy}{dx} = \frac{1}{\frac{df^{-1}}{dy}(y)}$$

ASSUME $f(x)$ IS NICE

$$(f \circ f^{-1})(x) = x \quad \Big| \ \frac{d}{dx}$$

CHAIN RULE:

$$\left(\frac{d}{dx} f\right)(f^{-1}(x)) \cdot \frac{df^{-1}}{dx}(x) = 1$$

$$\frac{df^{-1}}{dx}(x) = \frac{1}{\frac{df}{dx}(f^{-1}(x))}$$

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

EX: $f(x) = e^x$ DOMAIN: $(-\infty, +\infty)$

RANGE: $(0, +\infty)$

$f^{-1}(x)$ DOMAIN: $(0, +\infty)$ $\quad f'(x) = e^x$

RANGE: $(-\infty, +\infty)$

$f^{-1}(x) = \ln x$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$(\ln x)' = \frac{1}{x}$$

EX: $f(x) = \log_b x$, $b > 0$

$$\log_b x = \frac{\ln x}{\ln b}$$

$$f'(x) = \left(\frac{\ln x}{\ln b}\right)' = \frac{1}{\ln b}(\ln x)' = \frac{1}{x \ln b}$$

EX: $f(x) = \ln(g(x))$

$$f'(x) = \frac{1}{g(x)} \cdot g'(x) \qquad \boxed{f' = \frac{g'}{g}}$$

EX: $f(x) = \ln(1+x^2)$

$$f'(x) = \frac{(1+x^2)'}{1+x^2} = \frac{2x}{1+x^2}$$

SIMPLIFY WHEN POSSIBLE

EX: $f(x) = \ln(1000x)$

$$f'(x) = \frac{(1000x)'}{1000x} = \frac{\cancel{1000}}{\cancel{1000}x} = \frac{1}{x}$$

$$\ln(1000x) = \ln 1000 + \ln x$$

$$[\ln(1000x)]' = (\ln 1000 + \ln x)' = (\ln x)' = \frac{1}{x}$$

EX: $f(x) = \ln \frac{x^2}{\sqrt{x+7}}$ $\quad x > 0$

$$f'(x) = \frac{\left(\frac{x^2}{\sqrt{x+7}}\right)'}{\frac{x^2}{\sqrt{x+7}}} = \ldots$$

$$f(x) = \ln x^2 - \ln\sqrt{x+7} = 2\ln x - \frac{1}{2}\ln(x+7)$$

$$f'(x) = \frac{2}{x} - \frac{1}{2}\cdot\frac{1}{x+7}$$

$\ln(-3)$ DNE

RECALL: $|x| = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$

$\ln|-3| = \ln 3$

$\ln|x|$ IS WELL-DEFINED WHEN $x \neq 0$

CHECK: $\frac{d}{dx} \ln|x| = \frac{1}{x}$ FOR $x \neq 0$

LOGARITHMIC DIFFERENTIATION

$y = f(x) \qquad \frac{dy}{dx} = ?$

IF $y > 0$

$\ln y = \ln f(x) \ \Big| \frac{d}{dx}$

$$\frac{\frac{dy}{dx}}{y} = \frac{d}{dx}[\ln f(x)]$$

$$\frac{dy}{dx} = y \frac{d}{dx}[\ln f(x)]$$

EX: $f(x) = b^x$, $b > 0$

$$y = b^x \quad | \ln$$

$$\ln y = \ln(b^x) = x \ln b \quad \left| \frac{d}{dx} \right.$$

$$\frac{\frac{dy}{dx}}{y} = \frac{d}{dx}(x \ln b) = \ln b$$

$$\frac{dy}{dx} = (\ln b) \cdot y$$

$$(b^x)' = (\ln b) \cdot b^x$$

EX: $f(x) = x^r$, $x > 0$, r REAL NUMBER

$$y = x^r \quad | \ln$$

$$\ln y = \ln(x^r) = r \ln x \quad \left| \frac{d}{dx} \right.$$

$$\frac{\frac{dy}{dx}}{y} = \frac{r}{x} \qquad \frac{dy}{dx} = \frac{r}{x} \cdot x^r = r x^{r-1}$$

EX: $f(x)=x^r$, $x<0$, ASSUMING x^r IS WELL-DEFINED

$r=\frac{1}{3}$, $r=-7$, $r=\frac{5}{9}$

$y=x^r$

IF NECESSARY, USE $|\ |$

$$|y|=|x^r|=|x|^r \quad \Big| \ln$$

$$\ln|y|=\ln|x|^r=r\ln|x| \quad \Big| \frac{d}{dx}$$

$$\frac{\frac{d}{dx}|y|}{|y|}=\frac{r}{x}$$

$x<0$

$y=x^r<0$

$|y|=-y$

$$\cancel{\frac{d}{dx}(-y)= \frac{d}{dx}(-x^r)}$$

$$\frac{d}{dx}(-y)=-\frac{dy}{dx}$$

$$\frac{+\frac{dy}{dx}}{+y}=\frac{r}{x} \qquad \frac{dy}{dx}=y\cdot\frac{r}{x}=rx^{r-1}$$

Ex: $f(x) = x^x \, , \, x > 0$

$y = x^x \;|\; \ln$

$\ln y = \ln x^x = x \ln x \;|\; \frac{d}{dx}$

$$\frac{\frac{dy}{dx}}{y} = \frac{d}{dx}(x \ln x) = \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$