RATES OF CHANGE

$y = f(x)$, PICK $x_1 < x_2$

$\Delta x = x_2 - x_1$ RATE OF CHANGE IN x

$\Delta y = y_2 - y_1$ RATE OF CHANGE IN y

$y_1 = f(x_1)$, $y_2 = f(x_2)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

AVERAGE RATE OF CHANGE OF y WITH RESPECT TO x ON $[x_1, x_2]$

FIX x_1, LET $x_2 \to x_1$, THEN $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_1) = \frac{dy}{dx}(x_1)$$

INSTANTANEOUS RATE OF CHANGE

<u>EX 1</u> (PHYSICS)

$s = f(t)$ = POSITION OF A PARTICLE (MOVING ON A LINE) AT TIME t

$$\frac{ds}{dt} = f'(t) = \text{(INSTANTANEOUS) VELOCITY}$$

$$\frac{d^2s}{dt^2} = f''(t) = \text{ACCELERATION}$$

$$s = f(t) = t^3 - 3t^2 + 5 \, , \quad t \geq 0$$

1) FIND VELOCITY, ACCELERATION AT ALL t

2) WHEN IS THE PARTICLE AT REST?

3) WHEN DOES THE PARTICLE MOVE FORWARD?

4) WHAT IS THE TOTAL DISTANCE THE PARTICLE TRAVELS IN THE FIRST ~~TWO~~ FIVE SECONDS?

1) $f'(t) = 3t^2 - 6t$ VELOCITY

$f''(t) = 6t - 6$ ACCELERATION

2) SOLVE $f'(t) = 0$

$3t^2 - 6t = 0 \quad t(3t-6) = 0$ $t = 0$ OR $3t - 6 = 0$

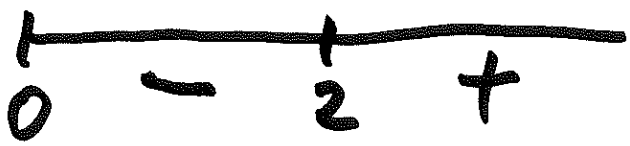
$t_1 = 0 \, , \; t_2 = 2$

~~2) WHEN DOES THE~~

3) PARTICLE MOVES FORWARD WHEN

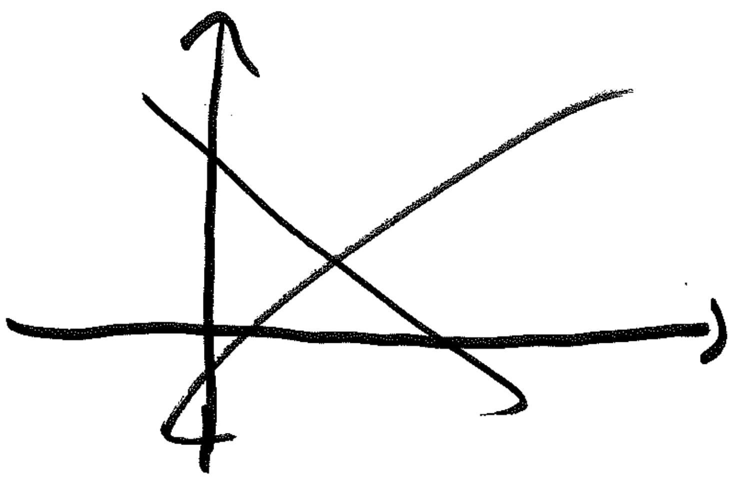
$f'(t) > 0$, $t > 2$

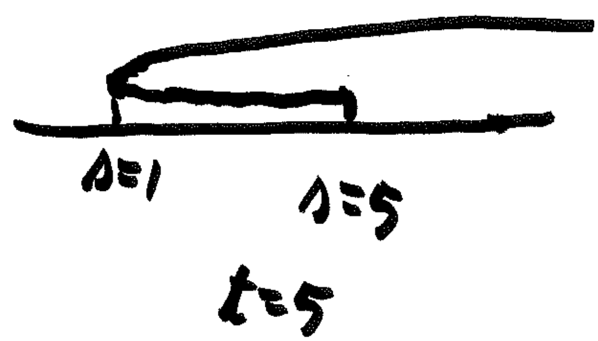
$t(3t-6) > 0$



$0 < t < 2$: $t > 0$, $3t-6 < 0$

$2 < t$: $t > 0$, $3t-6 > 0$





$t=2$, $s = f(2) = 2^3 - 3\cdot 2^2 + 5 = 1$ $\quad f(5) = 5^3 - 3\cdot 5^2 + 5 = 55$

4) NEED TO COMPUTE SUM OF DISTANCES TRAVELED TO THE LEFT AND RIGHT

BETWEEN 0 AND 2: $|f(2) - f(0)| = |1 - 5| = 4$

BETWEEN 2 AND 5: $|f(5) - f(2)| = |55 - 1| = 54$

TOTAL DIST: $4 + 54 = 58$

EX 2 (BIOLOGY)

$n = f(t)$ POPULATION OF SHARKS AT TIME t

LOOK AT TIMES $t=0, t=1, t=2, \ldots$

INSTANTANEOUS RATE OF GROWTH : $\frac{dn}{dt} = f'(t)$

ASSUME THAT THE INITIAL SHARK POPULATION IS 400, AND IT DOUBLES EVERY YEAR. WHAT IS THE POPULATION AFTER 3.5 YEARS?

$$f(0) = 400$$

$$f(1) = 400 \cdot 2$$

$$f(2) = 400 \cdot 2 \cdot 2 = 400 \cdot 2^2$$

$$f(3) = 400 \cdot 2 \cdot 2 \cdot 2 = 400 \cdot 2^3$$

$$f(4) = 400 \cdot 2^4$$

$$\vdots$$

$$f(n) = 400 \cdot 2^n$$

$f(t) = 400 \cdot 2^t$ FOR ALL $t > 0$

$$f(3.5) = 400 \cdot 2^{3.5}$$

WHAT IS THE INSTANTANEOUS GROWTH?

$$f'(3.5) = 400 \cdot 2^{3.5} \ln 2$$

$$f'(t) = (400 \cdot 2^{t})' = 400 \cdot 2^{t} \ln 2$$

EX 3 (ECONOMY)

$C(x)$ = COST OF PRODUCING x DOUGHNUTS

MARGINAL COST OF PRODUCING ONE EXTRA DOUGHNUT IS $C(x+1) - C(x)$

PASS TO THE LIMIT : MARGINAL COST

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

IF x IS VERY LARGE, $C(x+1) - C(x) \approx C'(x)$

$$C(x) = 500 + 2x + 5x^2$$

WHAT IS THE COST OF PRODUCING THE 43RD DOUGHNUT?

$$C(43) - C(42) = \ldots$$

OR: $C(43) - C(42) \approx C'(42)$

$C'(x) = 2 + 10x$

$C'(42) = 2 + 10 \cdot 42 = 422$

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e AS A LIMIT

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$f'(1) = 1$

$$1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \ln(1+h)$$

$$= \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}}$$

$$\lim_{h \to 0} \ln(1+h)^{\frac{1}{h}} = \ln e$$

$$\boxed{\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e}$$