EXPONENTIAL GROWTH/DECAY

BIOLOGY: $P(t)$ = POPULATION OF SHARKS AT TIME t

NO PREDATORS: $\frac{dP}{dt} \approx P$

$\frac{dP}{dt} = kP$ $\quad$ k = RELATIVE GROWTH RATE

$\frac{dP}{dt} = P$ $\quad$ GUESS $P(t) = e^t$

$(2e^t)' = 2(e^t)' = 2e^t$

MORE GENERALLY, $P(t) = Ce^t$ FOR ANY CONSTANT C

$(e^{kt})' = e^{kt} \cdot (kt)' = ke^{kt}$

$P(t) = Ce^{kt}$ SOLVE $\frac{dP}{dt} = kP$

EX: INITIALLY THERE 100 SHARKS, AND THE ~~GROW~~ RELATIVE GROWTH RATE IS $k=0.5$. HOW MANY SHARKS ARE THERE AFTER 5 YEARS?

$P(t)=Ce^{kt}$ $\quad k=0.5$

$P(0)=100=C\cdot e^{k\cdot 0}$, $C=100$

$P(t)=100\cdot e^{0.5t}$

$P(5)=100\cdot e^{2.5}$

EX: AFTER 1 YEAR, THERE ARE 5 SHARKS

AFTER 3 YEARS, THERE ARE 10 SHARKS

AFTER 9 YEARS, HOW MANY SHARKS?

$P(t)=Ce^{kt}$ $\quad P(1)=5$, $P(3)=10$

$5=P(1)=Ce^{k}$

$10=P(3)=Ce^{3k}$

DIVIDE

$\frac{10}{5}=\frac{\cancel{C}e^{3k}}{\cancel{C}e^{k}}$ $\quad 2=e^{2k}$ $\quad e^{k}=\sqrt{2}$

$k=\ln\sqrt{2}$

$$5 = Ce^{k} = C\cdot\sqrt{2}\ ,\quad C = \frac{5}{\sqrt{2}}$$

$$P(t) = \frac{5}{\sqrt{2}}\cdot\left(e^{k}\right)^{t} = \frac{5}{\sqrt{2}}\cdot\left(\sqrt{2}\right)^{t}$$

$$P(9) = \frac{5}{\sqrt{2}}\cdot\left(\sqrt{2}\right)^{9} = \frac{5}{\sqrt{2}}\cdot 2^{\frac{9}{2}} = 5\cdot 2^{\frac{9}{2}-\frac{1}{2}}$$

$$= 5\cdot 2^{4} = 80$$

HALF-LIFE OF RADIOACTIVE MATERIALS

$m(t)$ = MASS OF MATERIAL

$$\frac{dm}{dt} = km\ ,\quad k<0$$

HALF-LIFE = TIME THAT IT TAKES FOR m TO REDUCE TO HALF

EX: HALF-LIFE IS 1000 YRS

$$m(1000) = \frac{1}{2}m(0)$$

$$m(t) = m(0)\cdot e^{kt}$$

$$m(1000) = m(0)\cdot e^{1000k} \qquad \frac{1}{2}\cancel{m(0)} = \cancel{m(0)}\cdot e^{1000k}$$

$$\frac{1}{2} = e^{1000k}$$

$\ln \frac{1}{2} = 1000k$

$$k = \frac{\ln \frac{1}{2}}{1000} = -\frac{\ln 2}{1000}$$

EX (NEWTON'S LAW OF COOLING)

$T(t)$ = TEMPERATURE AT TIME t OF THE OBJECT

T_e = TEMPERATURE OF THE ENVIRONMENT

GUESS: $\lim_{t \to \infty} T(t) = T_e$

$$\frac{dT}{dt}(t) = k\underbrace{(T(t) - T_e)}_{y(t)}, \quad k < 0$$

$$\frac{dy}{dt} = \frac{d}{dt}(T(t) - T_e) = \frac{dT}{dt} - 0 = \frac{dT}{dt}$$

$$\frac{dy}{dt} = ky \qquad y(t) = Ce^{kt}$$

$$T(t) = T_e + Ce^{kt}$$

$$\lim_{t \to \infty} T(t) = T_e$$

Ex: $T(0)=100$, $T_e=50$

$T(2)=70$

Q: $T(6)=?$

$$T(t)=T_e+Ce^{kt}$$

$$100=T(0)=50+Ce^{k\cdot 0}=50+C, \quad C=50$$

$$70=T(2)=50+Ce^{2k}$$

$$70=50+50e^{2k}$$

$$20=50e^{2k} \qquad e^{2k}=\frac{2}{5}$$

$$e^{k}=\sqrt{\frac{2}{5}} \qquad k=\ln\sqrt{\frac{2}{5}}<0$$

$$T(t)=50+50\cdot\left(\sqrt{\frac{2}{5}}\right)^{t}$$

$$T(6)=50+50\cdot\left(\sqrt{\frac{2}{5}}\right)^{6}=50+50\cdot\left(\frac{2}{5}\right)^{3}=\ldots$$

ECONOMICS

COMPUTING INTEREST

$y(t)$ = AMOUNT OF MONEY IN BANK ACCOUNT AT TIME t (YEARS)

$y(0) = 5000$

r = INTEREST RATE = 4%

$y(3) = ?$

YEARLY INTEREST RATE

$y(1) = 5000 + 5000 \cdot r = 5000(1+r)$

$$y(2) = 5000(1+r) + 5000 \cdot (1+r) \cdot r$$
$$= 5000(1+r)^2$$

$y(3) = 5000(1+r)^3$

$y(t) = 5000(1+r)^t$

DAILY INTEREST RATES

$$y(1) = 5000\left(1 + \frac{r}{365}\right)^{365}$$

$$y(t) = 5000\left(1 + \frac{r}{365}\right)^{365t}$$

CONTINUOUS INTEREST RATE

$$y(t) = 5000\left(1+\frac{r}{n}\right)^{nt}$$

$$\lim_{n\to\infty} y(t) = ?$$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = \lim_{h\to 0}(1+h)^{\frac{r}{h}}$$

$$h = \frac{r}{n}\ ,\ n = \frac{r}{h}$$

$$= \lim_{h\to 0}\left[(1+h)^{\frac{1}{h}}\right]^{r} = e^{r}$$

$$\lim_{n\to\infty} y(t) = 5000 \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{n}\right)^{t}$$

$$= 5000\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t} = 5000\left(e^{r}\right)^{t}$$

$$= 5000\, e^{rt}$$

$\underline{EX}$: AFTER 3 YRS., $y(3) \approx 5000 \cdot e^{3 \cdot .04}$