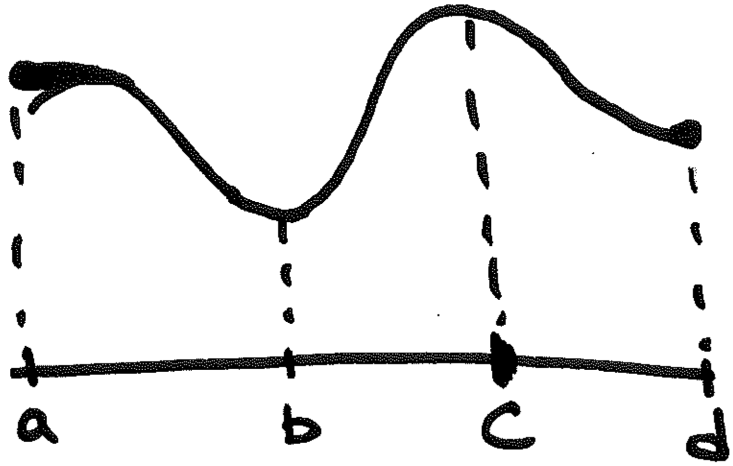
Maximum and Minimum values

Let f be a function with domain D.

<u>Def</u>: We say f has a <u>global maximum</u> (minimum) at point a if $f(a) \geq f(x)$ (respectively $f(a) \leq f(x)$) for all x in D.

We say f has a <u>local maximum</u> (min) at pt a if $f(a) \geq f(x)$ (respectively $f(a) \leq f(x)$) for all x near a.

Ex:



Local max at a

Global min at b

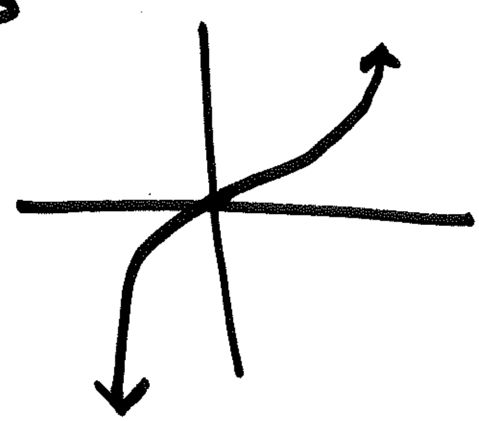
Global max at c

Local min at d

Is a global max also a local max? Yes! Always true!

Vise vera? NO! Not always true!

Ex: $f(x) = x^3$



No Max

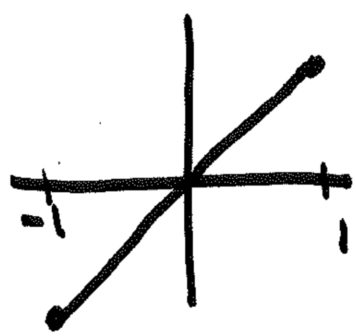
No min

When are we guaranteed to have extreme values?

Extreme Value Theorem:

If f is continuous on a closed interval $[a,b]$, then f has a **global max** and **global min**, these can occur at the endpoints.

Ex: 1

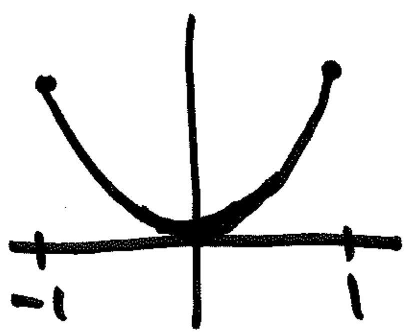


$f(x) = x$ on $[-1,1]$

global max occurs at $x=1$

global min occurs at $x=-1$

2

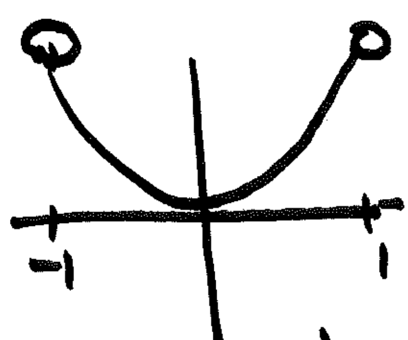


$f(x) = x^2$ on $[-1,1]$

global max at $x = \pm 1$

global min at $x=0$

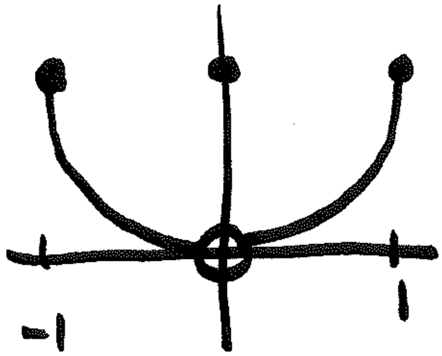
3



$f(x) = x^2$ on $(-1,1)$

global min at $x=0$

(open interval) $\rightarrow$ NO global max

(4) 

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

on $[-1, 1]$

Global max at $x = \pm 1, 0$. No global min → Not continuous

<u>Fermat's theorem</u>: If f has a local extrema at $a < c < b$ **AND** $f'(c)$ exists then

$f'(c) = 0$.

Why? <u>Pf</u>: Suppose f has a local min at c. For [redacted] small h, $h > 0$ → $f(c) \le f(c+h)$.

$-f(c) \qquad -f(c)$

$0 \le f(c+h) - f(c)$

$$f'(c) = \lim_{h \to 0^+} \left(\frac{f(c+h) - f(c)}{h} \right) \ge 0 \quad \text{b/c } \frac{+}{+}$$

Similarly for $h < 0$, h small,

$f(c) \le f(c+h)$

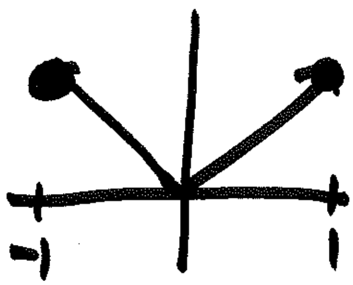
$$f'(c) = \lim_{h \to 0^-} \left(\frac{f(c+h) - f(c)}{h} \right) \leq 0$$

b/c $\frac{+}{-}$

Thus $f'(c) = 0$.

<u>Remark</u>: It is possible to have local extrema also if $f'(c)$ DNE <u>or</u> at the endpoints.

<u>Ex</u>:



$f(x) = |x|$

global min at $x=0$

($f'(x)$ DNE)

global max at endpoints

<u>Definition</u>: An interior pt $a < c < b$ is called a <u>critical point</u> if $f'(c) = 0$ or $f'(c)$ DNE.

How to find global extrema of continuous $f:[a,b] \to \mathbb{R}$.

(1) Find all critical points of $f(x)$: $c_1, c_2, \ldots$

(2) Find max/min by testing/comparing

crit pts $\to f(c_1), f(c_2), \ldots$

endpts $\to f(a), f(b)$

Ex 1 Find global extrema of $f(x) = x^2 - 2x + 5$ on $D = [-3,3]$.

(1) crit pts $f'(x) = 2x - 2$

$$2x - 2 = 0$$

$$\frac{2x = 2}{2} \Rightarrow x = 1$$

(2) Test

$$f(1) = 1^2 - 2(1) + 5 = 4$$

$$f(-3) = (-3)^2 - 2(-3) + 5 = 20$$

$$f(3) = 8$$

Global min 4 occurs at $x = 1$

Max = 20 occurs at $x = -3$

Ex 2: $f(x) = \sqrt[3]{x}$ on $D = [-1, 3]$

$= x^{1/3}$

① crit pts

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$$

crit pt → $x = 0$

② Test

$f(-1) = -1$ Min

$f(3) = \sqrt[3]{3}$ Max

$f(0) = 0$