MEAN VALUE THEOREM

Q1: HOW MANY SOLUTIONS DOES
$x^5+3x^3+x+4=0$ HAVE?

Q2: IF $f'=g'$, HOW ARE f AND g RELATED?

Q3: THE TURTLE AND THE HARE START AND FINISH AT THE EXACT SAME TIME. SHOW THAT THEY HAD THE SAME SPEED AT AN INTERMEDIATE TIME

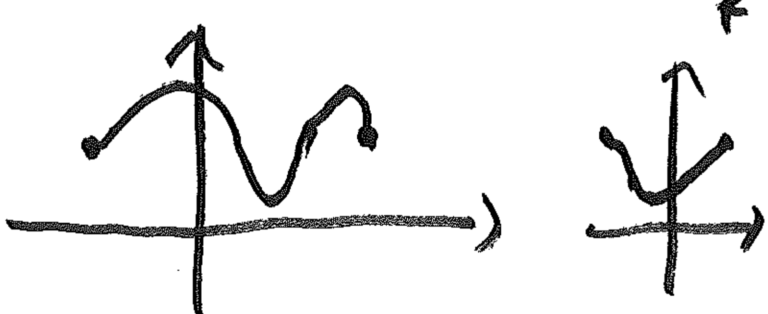
ROLLE'S THEOREM: ASSUME

i) f CONTINUOUS ON $[a,b]$

ii) f DIFFERENTIABLE ON (a,b)

iii) $f(a)=f(b)$

THEN THERE IS $a<c<b$ SO THAT
$f'(c)=0$



<u>PROOF</u>: f MUST HAVE BOTH A GLOBAL MINIMUM AND A GLOBAL MAXIMUM BY i)

a) IF BOTH GLOBAL MAX. AND MIN. ARE ACHIEVED AT a

THEN $f(x) \leq f(a)$ AND $f(x) \geq f(a)$

SO $f(x) = f(a)$ FOR ALL $a < x < b$

THEN $f'(x) = 0$ FOR ALL $a < x < b$; PICK ANY c

b) SAY THE GLOBAL MAXIMUM IS ACHIEVED AT $a < c < b$; THEN $f'(c) = 0$ BY FERMAT

<u>EX 1</u>: $-1 \leq x \leq 1$, $f(x) = 1 - x^2$

$f(-1) = 0$ THERE IS $-1 < c < 1$ SO
$f(1) = 0$ THAT $f'(c) = 0$

$f'(x) = -2x$, SO $c = 0$

<u>EX 2</u>: $-1 \leq x \leq 1$, $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

$f(-1) = f(1) = 1$

$f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ $f'(0)$ DNE

CONDITION ii) NOT TRUE

EX 3: $\underbrace{x^5+3x^3+x+4}_{f(x)}=0$ EXACTLY ONE SOLUTION

$f(-1)=(-1)^5+3\cdot(-1)^3+(-1)+4=-1$

$f(1)=1+3+1+4=9$

BY IVT, THERE IS $-1<c<1$ SO THAT $f(c)=0$

$f'(x)=5x^4+9x^2+1>0$

SAY $c_1<c_2$ SO THAT $f(c_1)=f(c_2)=0$

BY ROLLE'S THEOREM, THERE IS $c_1<c_3<c_2$ SO THAT $f'(c_3)=0$, IMPOSSIBLE

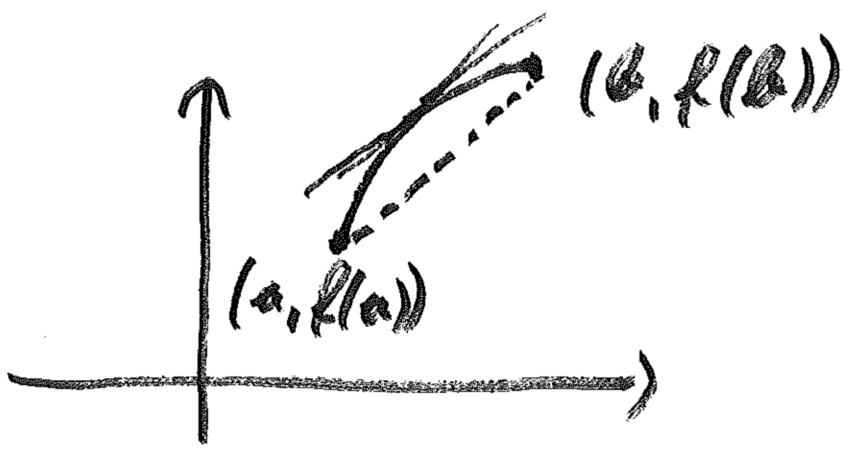
<u>MVT</u>: ASSUME

i) f CONTINUOUS ON $[a,b]$

ii) f DIFFERENTIABLE ON (a,b)

THEN THERE IS $a<c<b$ SO THAT

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



PROOF: LET $g(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$

$$g(a)=f(a)+\frac{f(b)-f(a)}{b-a}(a-a)=f(a)$$

$$g(b)=f(a)+\frac{f(b)-f(a)}{\cancel{b-a}}\cancel{(b-a)}=f(b)$$

LOOK AT $h(x)=f(x)-g(x)$

$$h(a)=f(a)-f(a)=0$$

$$h(b)=f(b)-f(b)=0$$

BY ROLLE, THERE IS $a<c<b$ SO THAT $h'(c)=0$

THUS $f'(c) = g'(c) = \frac{h(b)-h(a)}{b-a}$

Q3 $f(t)$ = POSITION OF HARE AT TIME t

$g(t)$ = POSITION OF TURTLE AT TIME t

START AT $t=0$, FINISH AFTER b HOURS

$f(0) = g(0)$

$f(b) = g(b)$

$h(t) = f(t) - g(t)$

$h(0) = h(b) = 0$

BY ROLLE'S THEOREM, THERE IS $0 < c < b$ SO THAT $h'(c) = 0$

$h'(c) = f'(c) - g'(c) = 0$, $f'(c) = g'(c)$

Q2 : IF $f'(x) = 0$ FOR ALL $a < x < b$, THEN f IS CONSTANT

PICK $a < x_1 < x_2 < b$, THEN $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0$

$f(x_2) = f(x_1)$

$f'(x) = g'(x)$ FOR ALL $a < x < b$

$h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) = 0$

$h(x) = C$, C CONSTANT

$f(x) = g(x) + C$

EX: $f'(x) = \underbrace{\cos x}_{(\sin x)'}$, $f(0) = 5$

$f(x) = \sin x + C$

$x = 0$: $5 = \sin 0 + C$, $C = 5$

$f(x) = \sin x + 5$