HOW DERIVATIVES AFFECT THE GRAPH OF A FUNCTION

RECALL: f INCREASING IF $f(x_1) < f(x_2)$ WHEN $x_1 < x_2$

f DECREASING IF $f(x_1) > f(x_2)$ WHEN $x_1 < x_2$

PROP: IF f DIFFERENTIABLE ON (a, b) AND:

i) $f'(x) > 0$ FOR ALL $a < x < b$, THEN f IS INCREASING

ii) $f'(x) < 0$ FOR ALL $a < x < b$, THEN f IS DECREASING

PROOF: PICK $a < x_1 < x_2 < b$, ASSUME i)

THEN $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \overset{MVT}{=} f'(c) > 0$

$f(x_2) - f(x_1) > 0 \quad f(x_1) < f(x_2)$

IN GENERAL, f WILL BE INCREASING ON SOME INTERVALS, DECREASING ON OTHERS

EX 1: $f(x) = x^3 + 3x - 1$

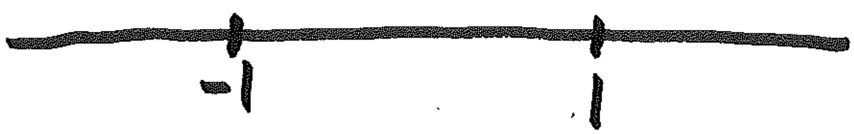
$f'(x) = 3x^2 + 3 > 0$

f IS INCREASING

EX 2: $f(x) = x^3 - 3x - 1$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

$f'(x) = 0$ AT $x = \pm 1$



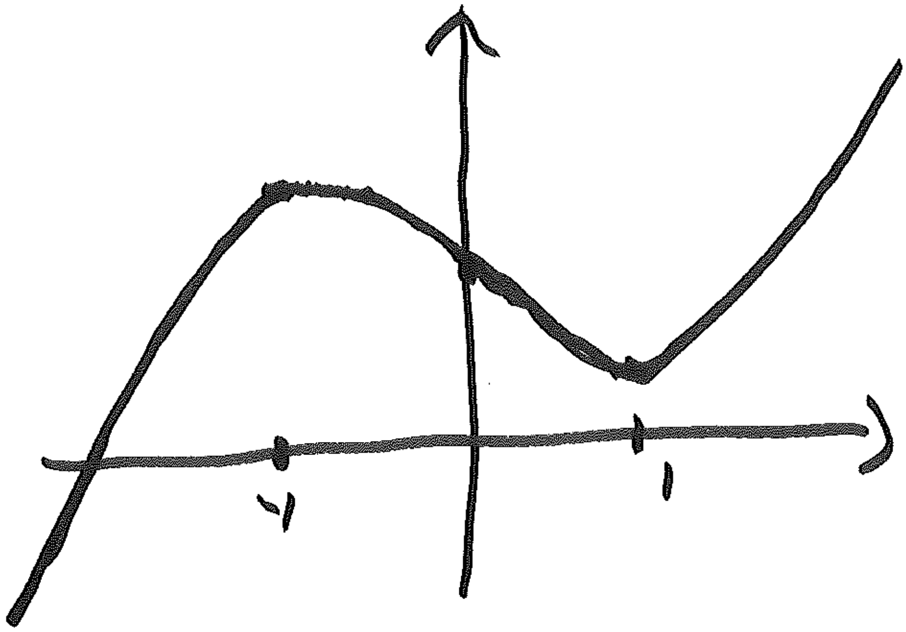
IF $x < -1$, $f'(x) = 3 \cdot (-) \cdot (-) > 0$

IF $-1 < x < 1$, $f'(x) = 3 \cdot (-) \cdot (+) < 0$

IF $1 < x$, $f'(x) = 3 \cdot (+) \cdot (+) > 0$

f INCREASING ON $(-\infty, -1)$ AND $(1, \infty)$

DECREASING ON $(-1, 1)$



FIRST DERIVATIVE TEST

ASSUME c IS A CRITICAL POINT OF f
THEN:

i) IF f' CHANGES SIGNS FROM $-$ TO $+$ AT c, THEN c IS A LOCAL MINIMUM

ii) IF f' CHANGES SIGNS FROM $+$ TO $-$ AT c, THEN c IS A LOCAL MAXIMUM

iii) IF f' DOES NOT CHANGE SIGN, THEN c IS NEITHER LOCAL MIN NOR LOCAL MAX

<u>Ex</u>: $f(x) = x^3 - 3x - 1$

$f'(x) = 3x^2 - 3$ $\quad x = \pm 1$ CRITICAL POINTS

$x = -1$: f' CHANGES FROM $+$ TO $-$, LOCAL MAX

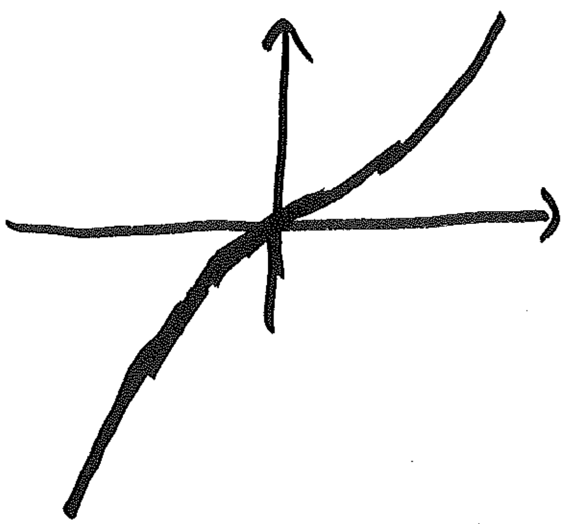
$x=1$: f' CHANGES FROM $-$ TO $+$, LOCAL MIN

EX: $f(x)=x^3$

$f'(x)=3x^2$, CRIT. POINTS $x=0$

$f'(x)>0$ WHEN $x<0$

$f'(x)>0$ WHEN $x>0$



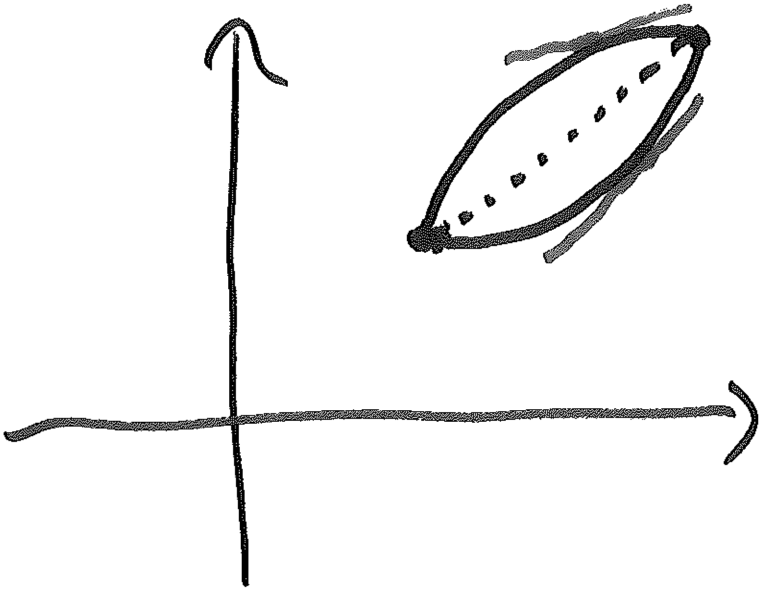
EX: $f(x)=|x|=\begin{cases}-x, & x<0\\ x, & x>0\end{cases}$

$f'(x)=\begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$, $f'(0)$ DNE

0 CRITICAL POINT

f' CHANGES FROM $-$ TO $+$ AT 0

$x=0$ IS A LOCAL MINIMUM



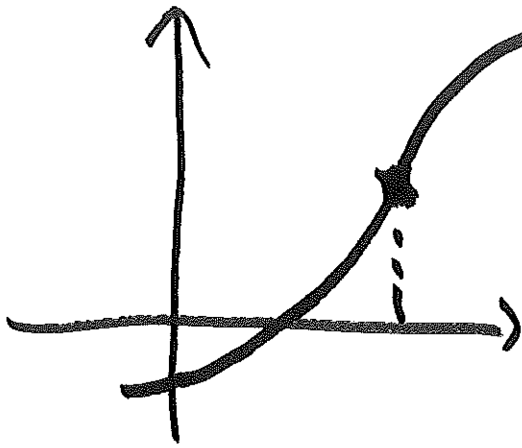
<u>DEF</u>: f IS CONCAVE UPWARDS IF THE GRAPH OF f LIES ABOVE ALL ITS TANGENTS

f IS CONCAVE DOWNWARDS IF THE GRAPH OF f LIES BELOW ALL ITS TANGENTS

<u>LEMMA</u>: IF $f''(x) > 0$ ON (a, b) THEN f IS CONCAVE UPWARDS ON (a, b)

IF $f''(x) < 0$ ON (a, b) THEN f IS CONCAVE DOWNWARDS ON (a, b)

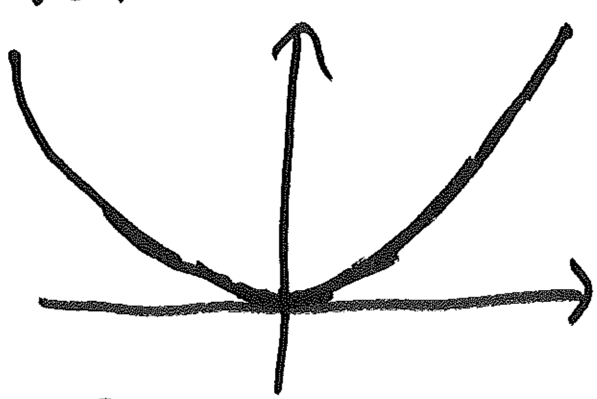
DEF: $(c, f(c))$ IS AN INFLECTION POINT IF f IS CONTINUOUS AT c, AND f CHANGES CONCAVITY THERE



LEMMA: IF c IS AN INFLECTION POINT, THEN $f''(c) = 0$ (OR DNE)

EX: $f(x) = x^2 \quad f'(x) = 2x, \; f''(x) = 2 > 0$

CONCAVE UPWARDS



EX: $f(x) = x^3 - 3x - 1 \quad f'(x) = 3x^2 - 3$

$f''(x) = 6x$

INFLECTION POINT $c=0$

IF $x<0$, $f''(x)<0$, CONCAVE DOWN

IF $x>0$, $f''(x)>0$, CONCAVE UP

SECOND DERIVATIVE TEST

IF $f'(c)=0$, f'' CONTINUOUS NEAR c

i) $f''(c)>0$, LOCAL MIN.

ii) $f''(c)<0$, LOCAL MAX.

iii) $f''(c)=0$, ??? DON'T KNOW

Ex: $f(x)=x^3-3x-1$

$f'(x)=3x^2-3$, $x=\pm 1$ CRIT. POINTS

$f''(x)=6x$

$f''(-1)=-6<0$ LOCAL MAX.

$f''(1)=6>0$ LOCAL MIN.