11/1 - L'Hôpital's Rule

- Use derivatives to compute limits like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Ex: $\lim_{x \to 1} \frac{\ln(x)}{x-1} \to \frac{\ln(1)}{1-1} = \frac{0}{0}$?

$\lim_{x \to \infty} \frac{x^3+1}{e^x} \Rightarrow \frac{\infty}{\infty}$?

L'Hôpital's Rule: If f, g diff'able near $x=a$ (except maybe at $x=a$) and either

i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ OR

ii) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$ (or $-\infty$)

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if this limit exists

- if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}$, can keep using L'Hôpital
- a can ~~keep~~ be $\pm\infty$

Ex 1: $\displaystyle \lim_{x\to 1} \frac{\ln(x)}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \frac{1}{1} = 1$

Ex 2: $\displaystyle \lim_{x\to\infty} \frac{x^3+1}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \to \frac{\infty}{\infty}$

$\displaystyle \overset{L'H}{=} \lim_{x\to\infty} \frac{6x}{e^x} \to \frac{\infty}{\infty}$

$\displaystyle \overset{L'H}{=} \lim_{x\to\infty} \frac{6}{e^x} \to \frac{6}{\infty} = \underline{0}$

Ex: $\displaystyle \lim_{x\to 0} \frac{e^x-1}{\sin(x)} \to \frac{1-1}{0} = \frac{0}{0}$

$\displaystyle \overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{\cos(x)} = \frac{1}{1} = 1$

Ex: $\displaystyle \lim_{x\to 0} \frac{e^x}{\sin(x)} = \frac{1}{0}$ DNE

$\to$ can't use L'Hôpital!

$$\lim_{x\to 0} \frac{e^x}{\sin(x)} \neq \lim_{x\to 0} \frac{e^x}{\cos(x)}$$

- L'Hôpital's Rule works for one sided limits

ex: $\lim_{x\to 0^+} \frac{1-\cos(x)}{\sqrt{x}} \to \frac{1-1}{\sqrt{0}} = \frac{0}{0}$ ✓

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{0-(-\sin(x))}{\frac{1}{2\sqrt{x}}}$ $\qquad \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \lim_{x\to 0^+} 2\sqrt{x}\sin(x) = 0$

- Can use L'Hôpital to evaluate "$0\cdot\infty$" limits by rewriting as "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

ex: $\lim_{x\to 0^+} x\ln(x) \to 0\cdot(-\infty)$ $\qquad x = \frac{1}{\frac{1}{x}}$

$\frac{-\infty}{\infty} \leftarrow \frac{\ln(x)}{\frac{1}{x}}$ $\qquad \frac{x}{\frac{1}{\ln(x)}}$

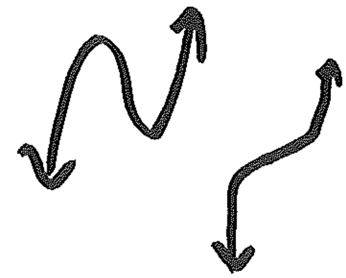
$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} -x = \underline{0}$

Sketching Graphs of Functions

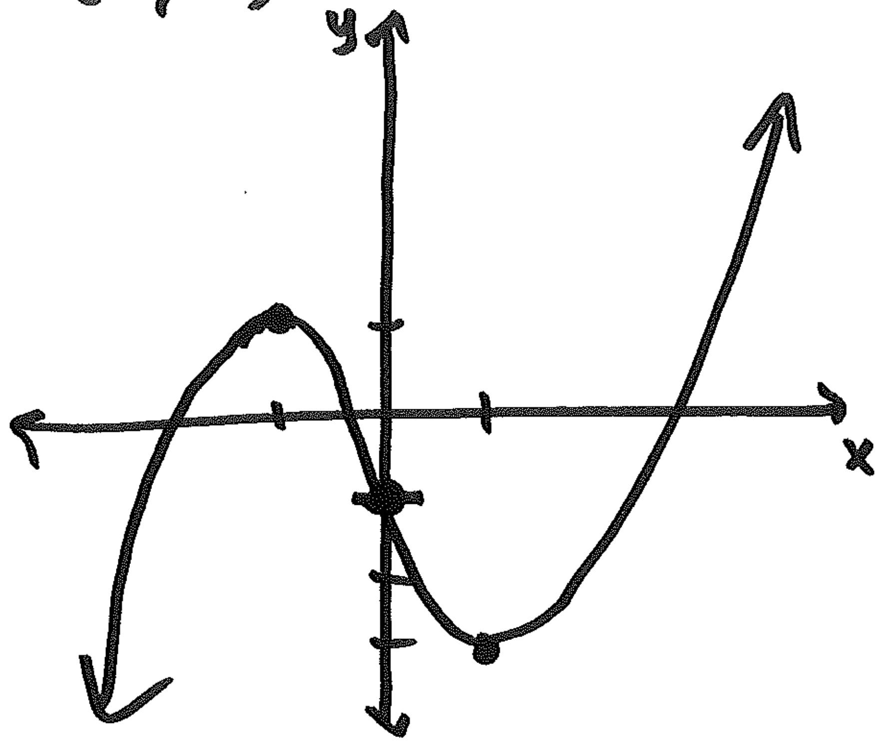
1) Find domain
2) Find y-intercept, x-intercepts if possible
3) Find f' and identify critical points
4) Find f'' and identify inflection points
5) Use $1^{st}/2^{nd}$ derivative tests to identify local max / local min
6) Identify concavity
7) Asymptotes?

Ex: $f(x) = x^3 - 3x - 1$



Domain: $(-\infty, \infty)$

y-int: $f(0) = -1$ $\quad (0,-1)$



$f'(x) = 3x^2 - 3$
$= 3(x^2 - 1)$
$= 3(x-1)(x+1)$

crit #s: $1, -1$

$f''(x) = 6x$

inflection pt: $x = 0$

1st deriv test: $\overset{+}{}\underset{-1}{|}\overset{-}{}\underset{\uparrow}{}\underset{1}{|}\overset{+}{}$ f'

- local max at $x=-1$
  $f(-1) = -1 + 3 - 1$
  $= 1$

$f'(-2) = 9 \qquad f'(0) = -3 \qquad f'(2) = 9$

- local min at $x=1$, $f(1) = 1 - 3 - 1 = -3$

concavity: $\overset{-}{}\underset{0}{|}\overset{+}{}$ f''

$\lim_{x\to\infty} f(x) = \infty$

concave down on $(-\infty, 0)$
concave up on $(0, \infty)$

$\lim_{x\to-\infty} f(x) = -\infty$