OPTIMIZATION

EX 1: FIND TWO NUMBERS WHOSE DIFFERENCE IS 10 AND PRODUCT AS SMALL AS POSSIBLE.

x, y TWO NUMBERS

$x - y = 10, \quad y = x - 10$

xy AS SMALL AS POSSIBLE

DEFINE $Q(x) = x(x-10) = x^2 - 10x$

LOOK AT CRITICAL POINTS OF Q

$Q'(x) = 2x - 10 \qquad Q'(x) = 0$ WHEN $x = 5$

$x = 5, \; y = -5 \qquad xy = 5 \cdot (-5) = -25$

LOCAL MINIMUM $\quad Q''(5) = 2 > 0$

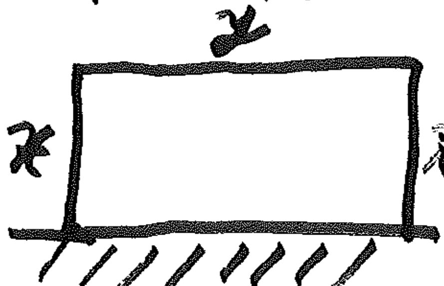
GENERAL STRATEGY

1) DRAW A PICTURE (IF NEEDED)
2) INTRODUCE VARIABLES, FUNCTION TO BE OPTIMIZED Q
3) WRITE DOWN RELATIONS BETWEEN VARIABLES

4) EXPRESS Q AS A FUNCTION OF ONE VARIABLE; WHAT IS THE DOMAIN OF Q?

5) FIND CRITICAL POINTS, FIND ABSOLUTE MIN (OR MAX)

<u>Ex 2</u>: ASSUME WE HAVE 40 FT OF FENCE, AND WANT TO ENCLOSE AS BIG OF AN AREA IN THE SHAPE OF A RECTANGLE, ASSUMING WE TAKE OUR HOUSE AS ONE OF THE SIDES



$2x+y=40$, $y=40-2x$

AREA $= xy$

$Q(x)=x(40-2x)=40x-2x^2$

$0 \leq x \leq 20$

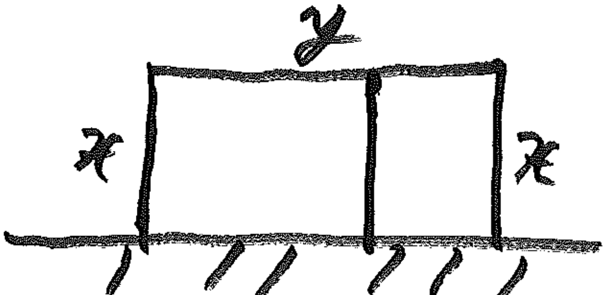
$Q'(x)=40-4x$ $\quad$ $Q'(x)=0$ WHEN $x=10$

$Q(10) = 10(40 - 2 \cdot 10) = 10 \cdot 20 = 200$

Q ACHIEVES ITS ABSOLUTE MAXIMUM ON $[0, 20]$ BY EVT

$$\underset{\underset{0}{\parallel}}{Q(0)},\ \underset{\underset{0}{\parallel}}{Q(20)},\ \underset{\underset{200\ \checkmark}{\parallel}}{Q(10)}$$ TAKE MAXIMUM

WHAT IF WE WANT TO SPLIT THE ENCLOSURE BY AN TWO WITH A FENCE PERPENDICULAR TO THE HOUSE?

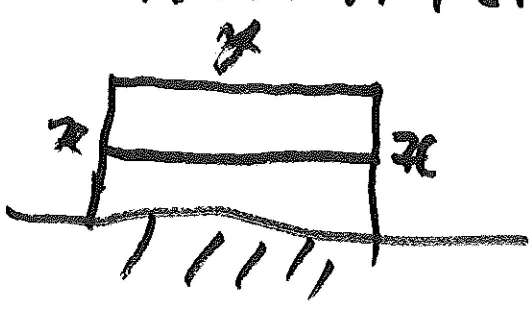


$3x + y = 40$, $y = 40 - 3x$

xy

$Q(x) = x \cdot (40 - 3x)$, FIND CRITICAL POINT

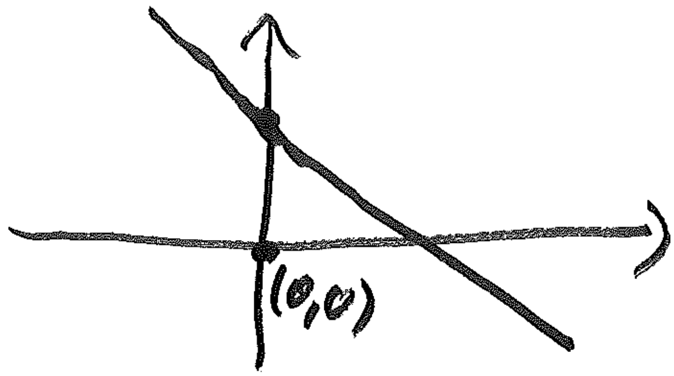
WHAT ABOUT A FENCE PARALLEL TO THE HOUSE?



$2x + 2y = 40$, $y = 20 - x$

$Q(x) = x(20 - x)$

EX 3: FIND THE MINIMUM DISTANCE BETWEEN THE ORIGIN $(0,0)$ AND THE LINE $y-2x=1$



$y-2x=1$, $y=2x+1$

WANT TO FIND MINIMUM OF $\sqrt{x^2+y^2}$

THIS HAPPENS AT THE SAME POINT AS WHERE THE MINIMUM OF x^2+y^2 IS

x^2+y^2 $\quad Q(x)=x^2+(2x+1)^2$

$$= x^2+4x^2+4x+1 = 5x^2+4x+1$$

$Q'(x)=10x+4$ $\quad Q'(x)=0$ $\quad 10x+4=0$, $x=-\frac{4}{10}$

$y=2\cdot\left(-\frac{4}{10}\right)+1=\frac{2}{10}$

$Q\left(-\frac{4}{10}\right)=\left(-\frac{4}{10}\right)^2+\left(\frac{2}{10}\right)^2=\frac{1}{5}$

A: $\sqrt{\frac{1}{5}}$ $\qquad$ DOMAIN OF Q: $(-\infty,+\infty)$

FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREMA

f DEFINED ON (a,b), c CRITICAL POINT

i) IF $f'(x)<0$ WHEN $x<c$ AND $f'(x)>0$ WHEN $x>c$ THEN c IS AN ABSOLUTE MINIMUM

ii) IF $f'(x)>0$ WHEN $x<c$ AND $f'(x)<0$ WHEN $x>c$ THEN c IS AN ABSOLUTE MAXIMUM

BACK TO EX 3: $Q'(x)=10x+4$ $\quad c=-\frac{4}{10}$

$x<-\frac{4}{10}$, $Q'(x)<0$

$x>-\frac{4}{10}$, $Q'(x)>0$

ABSOLUTE MINIMUM

EX 4: IF THE PRODUCT OF TWO POSITIVE NUMBERS IS 1, WHAT IS THEIR SMALLEST SUM?

$x, y \qquad xy=1 \qquad x+y$

$Q(x)=x+\frac{1}{x}$ $\qquad y=\frac{1}{x}$

$Q'(x)=1-\frac{1}{x^2}$ $\qquad c=1$

$Q'(x)<0$ WHEN $x<1$

$Q'(x)>0$ WHEN $x>1$