OPTIMIZATION

GENERAL STRATEGY:

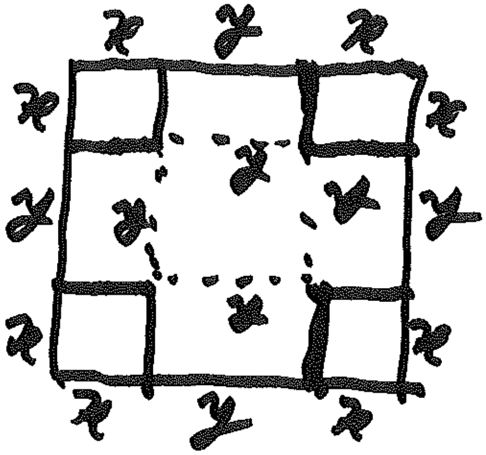
1) IF NECESSARY, DRAW A PICTURE
2) INTRODUCE VARIABLES, FUNCTION TO BE OPTIMIZED Q
3) WRITE DOWN RELATIONS BETWEEN VARIABLES
4) EXPRESS Q AS A FUNCTION OF <u>ONE</u> VARIABLE; FIND DOMAIN OF Q
5) TEST FOR CRITICAL POINTS, FIND <u>ABSOLUTE</u> EXTREMUM

IF DOMAIN IS $[a, b]$, USE EVT; COMPARE $Q(a)$, $Q(b)$ AND Q(CRITICAL PTS)

IF (a, b), a, b COULD BE $\pm\infty$

USE FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREMA

EX 1: BUILD AN OPEN-TOP BOX BY TAKING A 3x3 RECTANGULAR SHEET OF PAPER, CUTTING LITTLE SQUARES OUT OF THE CORNERS, AND FOLDING. WHAT IS THE MAXIMUM VOLUME?



$VOL = xy^2$

$2x + y = 3 \quad \therefore \quad y = 3 - 2x$

$x = \frac{3-y}{2}$

$Q(x) = x \cdot (3-2x)^2$

$Q(y) = \frac{3-y}{2} \cdot y^2, \quad 0 \leq y \leq 3$

$Q(x) = x(9 + 4x^2 - 12x) = 9x - 12x^2 + 4x^3$

$Q'(x) = 9 - 24x + 12x^2$, SOLVE $Q'(x) = 0 \ldots$

$Q(y) = \frac{3}{2}y^2 - \frac{y^3}{2}$

$Q'(y) = 3y - \frac{3}{2}y^2 = \frac{3}{2}y(2-3y)$

$y=0$ OR $y=\frac{2}{3}$

$Q(0)=0,\ Q(3)=0,\ Q\left(\frac{2}{3}\right)=\frac{3-\frac{2}{3}}{2}\cdot\left(\frac{2}{3}\right)^2$

$=\frac{7}{6}\cdot\frac{4}{9}=\frac{14}{27}$ ✓

EX 2:

DOG RUNS AT 3FT/S, SWIMS AT 1FT/S

WHAT IS THE MINIMUM TIME IT TAKES THE DOG TO REACH THE FRISBEE?

x = DISTANCE THE DOG RUNS

z = DISTANCE THE DOG SWIMS

y = DISTANCE ON LAND BETWEEN FRISBEE AND WHERE THE DOG STARTS SWIMMING

TIME = TIME ON LAND + TIME IN WATER

$=\frac{x}{3}+\frac{z}{1}$

$x+y=10$, $x=\cancel{y+10}\ 10-y$

$y^2+4^2=z^2$, $z=\sqrt{y^2+16}$

$$Q(y)=\frac{-y+10}{3}+\sqrt{y^2+16}$$

$$0\le y\le 10$$

$$Q'(y)=\frac{-1}{3}+\frac{1}{2\sqrt{y^2+16}}\cdot 2y \overset{?}{=} 0$$

$$-\frac{1}{3}+\frac{y}{\sqrt{y^2+16}}=0 \qquad \frac{y}{\sqrt{y^2+16}}=\frac{1}{3} \quad |(\ )^2$$

$$\frac{y^2}{y^2+16}=\frac{1}{9} \qquad 9y^2=y^2+16$$

$$8y^2=16,\ y^2=2,\ y=\sqrt{2}$$

$$\left.\begin{aligned} Q(\sqrt{2})&=\frac{-\sqrt{2}+10}{3}+\sqrt{18}\\ Q(0)&=\frac{10}{3}+4\\ Q(10)&=\sqrt{116}\end{aligned}\right\}\ Q(\sqrt{2})\text{ SMALLEST}$$

EX3 (ECON)

x DONUTS, $R(x)$ = REVENUE

$C(x)$ = COST

$P(x) = R(x) - C(x)$ PROFIT

GOAL: MAXIMIZE $P(x)$

$P'(x) = R'(x) - C'(x) \stackrel{?}{=} 0$

MARGINAL REVENUE = MARGINAL COST

SAY EACH DONUT COSTS \$2, COST OF MAKING x DONUTS IS $\frac{x^2}{800}$. HOW MANY DONUTS SHOULD YOU MAKE?

$R(x) = 2x$ $\qquad R'(x) = 2$

$C(x) = \frac{x^2}{800}$ $\qquad C'(x) = \frac{2x}{800} = \frac{x}{400}$

$R'(x) \stackrel{?}{=} C'(x)$ $\qquad 2 = \frac{x}{400}, \; x = 800$

$P(x) = 2x - \frac{x^2}{800}$ $\qquad$ DOMAIN $0 < x < +\infty$

$P'(800) = 0$ $\qquad P'(x) = 2 - \frac{x}{400}$

WHEN $0 < x < 800$, $P'(x) > 0$

$800 < x$, $P'(x) < 0$

BY FIRST DERIV. TEST, 800 IS AN ABSOLUTE MAXIMUM

EX: SELL 200 DONUTS A DAY AT \$2 EACH. FOR EACH 10¢ INCREASE, WE LOSE 5 CUSTOMERS. WHAT IS THE OPTIMAL PRICE FOR THE BUSINESS?

INCOME = # DONUTS SOLD · PRICE OF DONUT

PRICE OF DONUT $= 2 + \frac{x}{10}$

CUSTOMERS $= 200 - 5x$

$$Q(x) = \left(2 + \frac{x}{10}\right)(200 - 5x)$$

$$Q'(x) = \frac{1}{10}(200 - 5x) + \left(2 + \frac{x}{10}\right)\cdot(-5)$$

$$= 20 - \frac{5}{10}x - 10 - \frac{5}{10}x = 10 - x$$

CRIT. POINT $x = 10$

$Q(0) = 2 \cdot 200 = 400\$$

$Q(10) = \left(2 + \frac{10}{10}\right)\cdot(200 - 5\cdot 10)$
$= 3 \cdot 150 = 450\$$