WORD PROBLEMS:

RATES OF CHANGE
EXPONENTIAL GROWTH/DECAY
RELATED RATES
OPTIMIZATION

EX: ASSUME THAT THE SIDE OF A SQUARE INCREASES AT A RATE OF 6 m/s.

i) WHAT IS THERATE OF CHANGE OF THE AREA WITH RESPECT TO THE LENGTH OF THE SIDE L WHEN L=9m?

ii) WHAT IS THE AVERAGE RATE OF CHANGE WITH RESPECT TO L WHEN L CHANGES FROM 1m TO 3m?

iii) WHAT IS THE RATE OF CHANGE OF THE AREA WITH RESPECT TO TIME WHEN L=9m?

i) L = LENGTH

A = AREA

$A(L) = L^2$

$\frac{dA}{dL} = 2L$ WHEN $L=9$, $\frac{dA}{dL} = 2 \cdot 9 = 18$

ii) AVERAGE RATE OF CHANGE:

$$\frac{A(3)-A(1)}{3-1} = \frac{3^2-1^2}{2} = 4$$

iii) t = TIME

$\frac{dA}{dt} = ?$ WHEN $L=9$

$A = L^2 \quad \Big| \frac{d}{dt}$

$$\frac{dA}{dt} = \frac{d}{dt}(L^2) = 2L\frac{dL}{dt}$$

WHEN $L=9$: $\frac{dA}{dt} = 2 \cdot 9 \cdot 6 = 108 \ m^2/s$

SOLVE $\frac{dy}{dt} = ky$

$$y(t) = y(0)\,e^{kt}$$

<u>EX</u>: ASSUME THAT THE HALF-LIFE OF A ~~PARTICLE~~ SUBSTANCE IS 50 YEARS. IF WE START WITH 200 g, HOW MANY GRAMS ARE LEFT AFTER 70 YEARS?

$m(t)$ = MASS AT TIME t

$m(0) = 200$

$m(50) = \frac{1}{2}\cdot 200 = 100$

$m(t) = m(0)\,e^{kt}$

$t = 50$: $100 = 200\,e^{-k\cdot 50} \quad \Big|\ \frac{1}{100}\cdot e^{50k}$

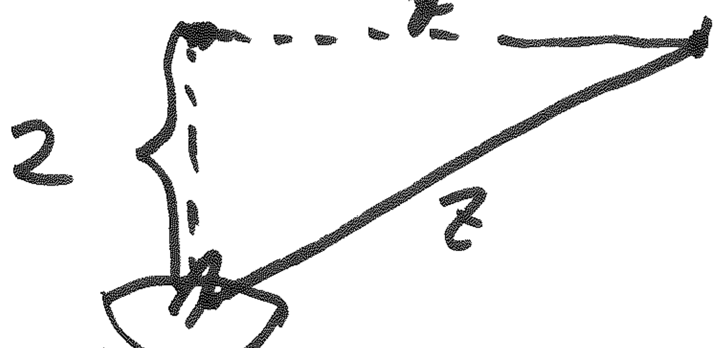
$$e^{50k} = 2 \qquad 50k = \ln 2,\ k = \frac{\ln 2}{50}$$

$$m(70) = 200\,e^{-70\cdot\frac{\ln 2}{50}} = 200\cdot e^{-\frac{7}{5}\ln 2}$$

$$= 200\cdot\left(e^{\ln 2}\right)^{-\frac{7}{5}}$$

$$= 200\cdot 2^{-\frac{7}{5}}$$

EX 3 : A PLANE FLIES HORIZONTALLY AT ALTITUDE OF 2 MILES, SPEED OF 400 MI/R. THE PLANE PASSES DIRECTLY OVER A RADAR STATION. AT WHAT RATE IS THE DISTANCE INCREASING (WITH RESPECT TO TIME) 30 MINUTES AFTER?



$z(t)$ = DISTANCE AT TIME t TO THE RADAR

$y(t)$ = HORIZONTAL DISTANCE

$\frac{dz}{dt}$ WHEN $t = 30$

BY PYTHAGORA'S, $z^2 = 2^2 + y^2 \;\Big|\; \frac{d}{dt}$

$\frac{d}{dt}(z^2) = 2z\frac{dz}{dt}$ $\quad \frac{d}{dt}(2^2 + y^2) = 2y\frac{dy}{dt}$

$$\cancel{2}z\frac{dz}{dt} = \cancel{2}y\frac{dy}{dt}$$

$$z\frac{dz}{dt} = y\frac{dy}{dt} \qquad \frac{dy}{dt} = 400$$

$$y(30\text{ MINS}) = y\left(\tfrac{1}{2}\right) = \frac{400}{2} = 200 \text{ mi}$$

$$z\left(\tfrac{1}{2}\right) = \sqrt{4 + 200^2}$$

$$\frac{dz}{dt} = \frac{200 \cdot 400}{\sqrt{4+200^2}} = \ldots$$

<u>EX 4</u>: FIND TWO POSITIVE NUMBERS x, y WHOSE SUM EQUALS 6 AND x^2y IS AS LARGE AS POSSIBLE

<u>KNOW</u>: $x + y = 6$, $y = 6 - x \geq 0$

$$Q(x) = x^2(6-x), \quad 0 \leq x \leq 6$$

OR

$$Q(y) = (6-y)^2 y$$

$Q'(x) = 12x - 3x^2$

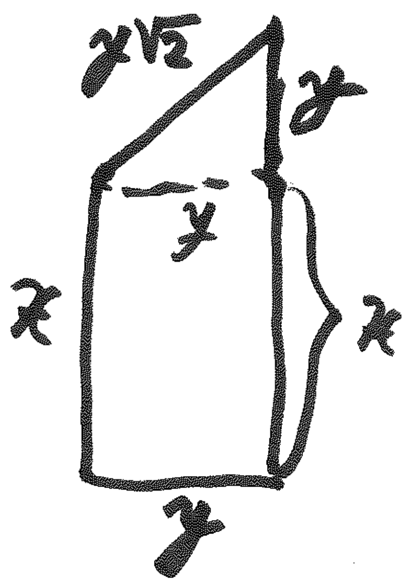
SOLVE $Q'(x) = 0$, $x = 0$ OR $x = 4$

BY EVT, MAX. EXISTS

$$\underset{0}{\underset{\shortparallel}{Q(0)}}, \underset{0}{\underset{\shortparallel}{Q(6)}}, \underset{32}{\underset{\shortparallel}{Q(4)}}$$

FIRST/SECOND DERIVATIVE TEST

<u>EX 5</u>: ASSUME WE NEED TO ENCLOSE AN AREA AS BELOW WITH 50 FT OF FENCE. WHAT IS THE MAXIMUM AREA?



$$\text{AREA} = xy + \frac{1}{2}y^2$$

$$50 = 2x + (2+\sqrt{2})y$$

$$2x = 50 - (2+\sqrt{2})y$$

$$x = 25 - \frac{2+\sqrt{2}}{2}y$$

$$Q(y) = \left(25 - \frac{2+\sqrt{2}}{2}y\right)y + \frac{1}{2}y^2$$

$$Q(y) = 25y - \frac{1+\sqrt{2}}{2} y^2$$

$$Q'(y) = 25 - (1+\sqrt{2})y \stackrel{?}{=} 0$$

$$y = \frac{25}{1+\sqrt{2}}$$

$$x = 25 - \frac{2+\sqrt{2}}{2} \cdot \frac{25}{1+\sqrt{2}} = 25 \cdot \frac{2-\sqrt{2}}{2}$$