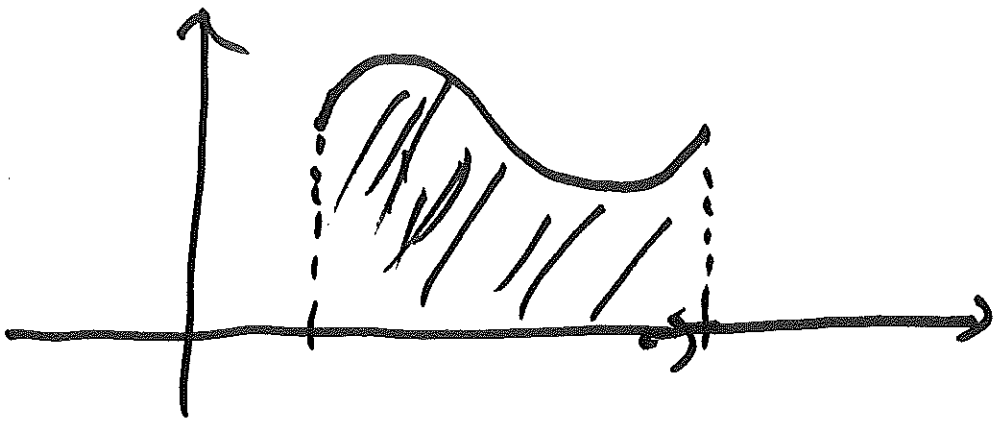
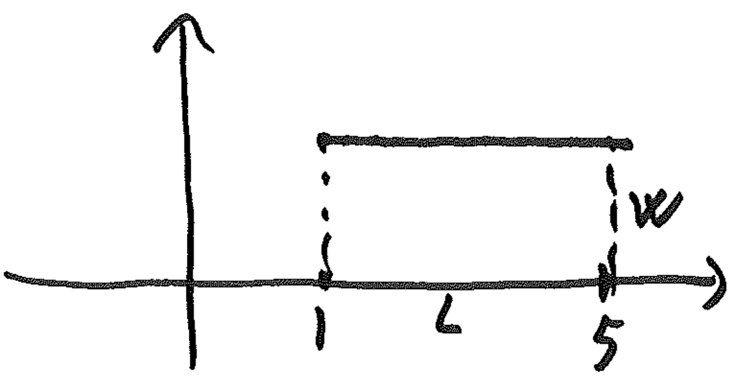
AREAS AND DISTANCES

Q: WHAT IS THE AREA UNDER THE GRAPH OF A FUNCTION?



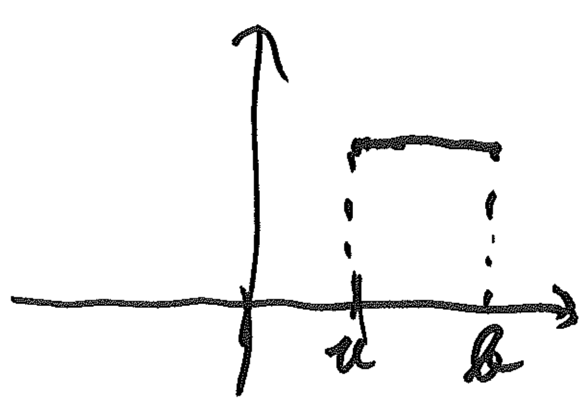
EX 1: $1 \le x \le 5$, $f(x) = 2$



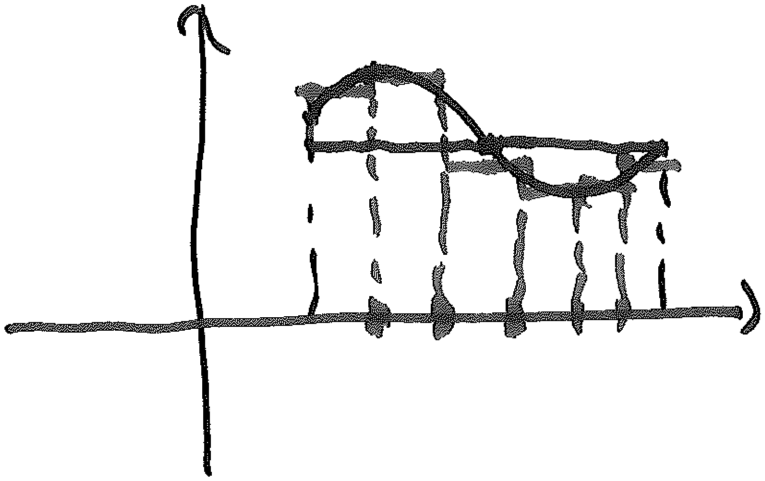
LENGTH $= 5 - 1 = 4$
WIDTH $= 2$

AREA $= 4 \cdot 2 = 8$

$a \le x \le b$, $f(x) = C$

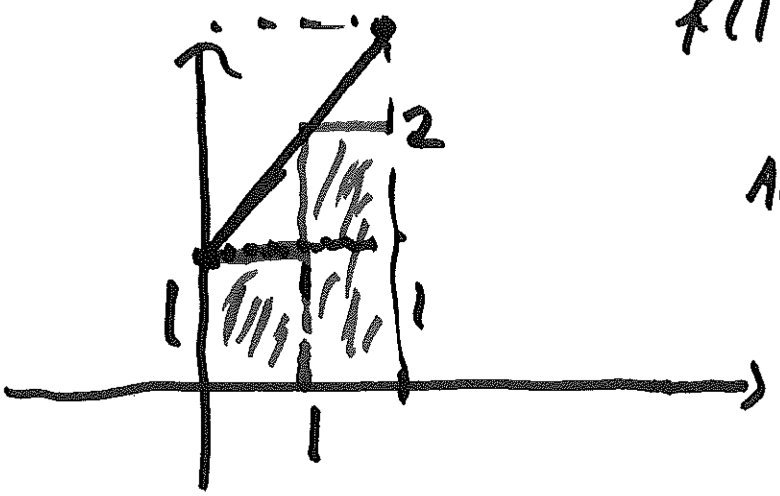


AREA $= C(b-a)$



EX 2 : $f(x) = 2x+1$, $0 \leq x \leq 1$

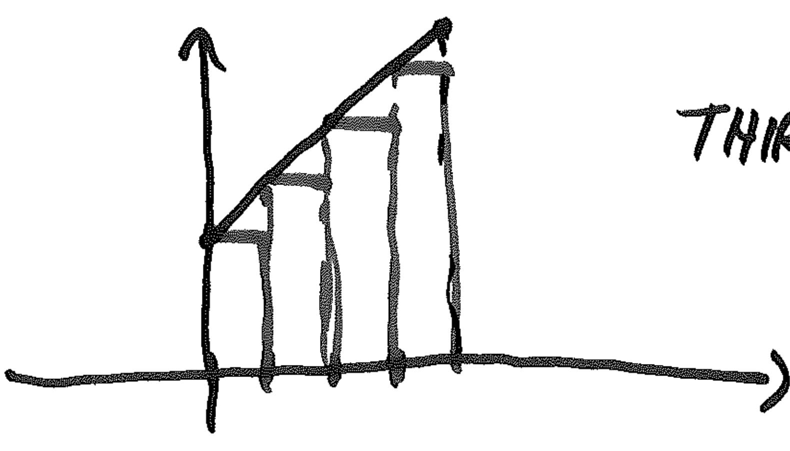
$f(1) = 3$



AREA $= 1 + \frac{1 \cdot 2}{2} = 2$

FIRST APPROX.: AREA $= 1$ OR AREA $= 3$

SECOND APPROX: AREA $= \frac{1}{2} + \frac{1}{2} \cdot 2 = \frac{1}{2} + 1 = \frac{3}{2}$



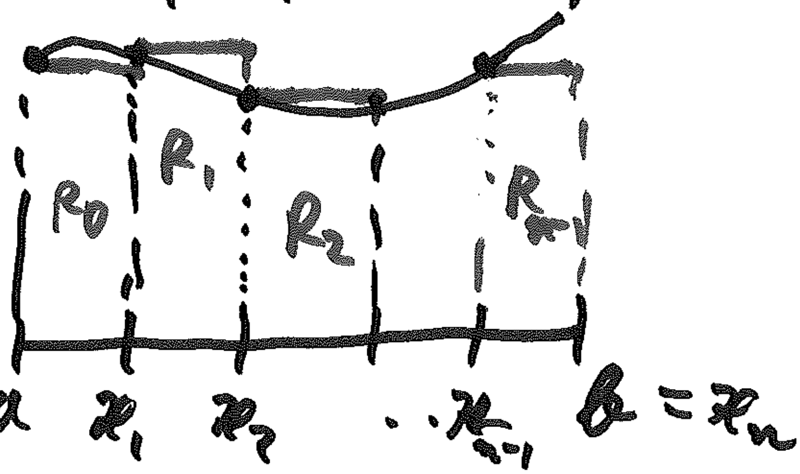
THIRD APPROX: CLOSER TO 2

IN GENERAL, $a \leq x \leq b$, $f(x) \geq 0$

APPROXIMATE THE AREA

i) DIVIDE $[a, b]$ INTO n EQUAL INTERVALS OF LENGTH $\Delta x = \frac{b-a}{n}$

$a = x_0$, $x_1 = a + \Delta x$; $x_2 = x_1 + \Delta x$; ... $x_n = b$



ii) LOOK AT $f(x_i)\Delta x$

$f(x_i)\Delta x$ IS THE AREA OF RECTANGLE R_i

LOOK AT $L_n = \Delta x \left(f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right)$

$$= \frac{b-a}{n} \cdot \sum_{i=1}^{n} f(x_{i-1})$$

iii) HOPE THAT $\lim_{n \to \infty} L_n = A$ (AREA)

SUMMATION NOTATION

GIVEN A FUNCTION f, DEFINE

$$\sum_{i=1}^{n} f(i) = f(1)+f(2)+\dots+f(n)$$

EX: $\displaystyle\sum_{i=1}^{4} i^2 = 1^2+2^2+3^2+4^2 = 1+4+9+16$

$= 30$

IMPORTANT SUMS

$$\sum_{i=1}^{n} 1 = 1+1+\dots+1 = n$$

$$\sum_{i=1}^{n} i = 1+2+\dots+n = \frac{n(n+1)}{2}$$

$$\underline{\quad n+(n-1)+\dots+1 \quad}$$

$$(n+1)+(n+1)+\dots+(n+1) = n(n+1)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

SEQUENCES: $L_1, L_2, \dots, L_n, \dots$

$L_i = f(i)$, $i \geq 0$ POSITIVE INTEGER

$\lim_{n\to\infty} L_n = A$ MEANS "L_n APPROACHES A" AS n GROWS TO ∞

IF $\lim_{x\to\infty} f(x) = A$, THEN IF $L_n = f(n)$,

$\lim_{n\to\infty} L_n = A$

Ex: $L_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} L_n = 0$

$f(x) = \frac{1}{x}$ $\quad \lim_{x\to\infty} f(x) = 0$

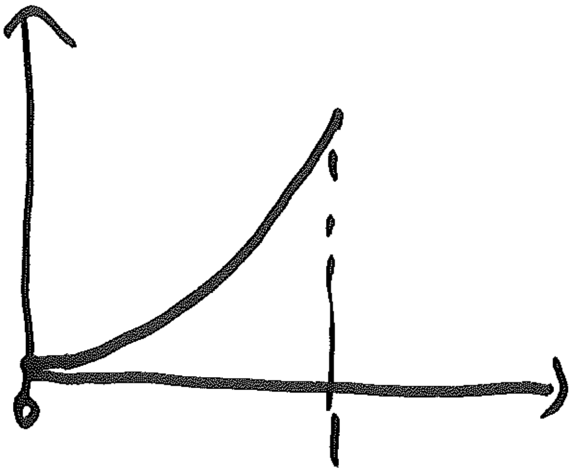
WE CAN PICK ALSO OTHER POINTS FOR THE HEIGHT OF THE RECTANGLE

$$R_n = \Delta x\left(f(x_1) + f(x_2) + \dots + f(x_n)\right)$$
$$= \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)$$

IN GENERAL, CAN PICK ANY $x_{i-1} \le x_i^* \le x_i$

$$S_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

Ex: $f(x)=x^2, \quad 0 \le x \le 1$



$R_n \qquad [0,1] \qquad x_0 = 0$

n PIECES $\qquad x_1 = \frac{1}{n},\ x_2 = \frac{2}{n}, \dots, x_n = \frac{n}{n}$

$\frac{1}{n} \qquad\qquad x_i = \frac{i}{n}$

$$R_n = \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2}$$

$$= \frac{1}{n}\cdot\frac{1}{n^2}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{x\to\infty}\frac{x(x+1)(2x+1)}{6x^3} \overset{\text{L'HOP}}{=} \frac{2}{6} = \frac{1}{3}$$

DISTANCE PROBLEM

ASSUMING WE KNOW THE VELOCITY AT ONLY CERTAIN TIMES, WHAT IS THE DISTANCE TRAVELED?

IF $v(t_i)$, $v(t_{i+1})$ ARE GIVEN

DIST. TRAVELED BETWEEN t_i, t_{i+1} BY

$v(t_i)(t_{i+1}-t_i)$ (OR $v(t_{i+1})(t_{i+1}-t_i)$)

EX: $v(0)=0$, $v(1)=3$, $v(2)=4$, $d(3)\approx ?$

$d(3)\approx 0\cdot(1-0)+3\cdot(2-1)+4\cdot(3-2)\approx 3+4=7$