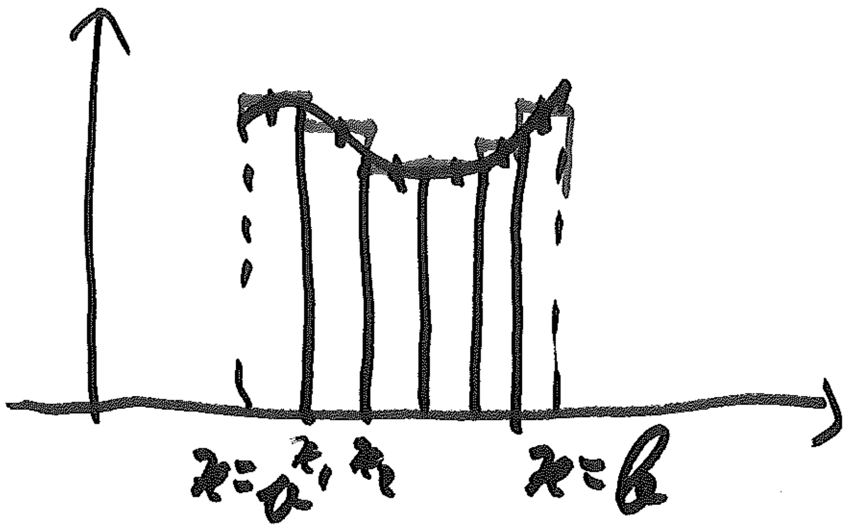
THE DEFINITE INTEGRAL



RIEMANN SUMS

f DEFINED ON $a \le x \le b$

i) PICK n POSITIVE INTEGER, LET $\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x \quad , \quad 0 \le i \le n$

ii) PICK $x_{i-1} \le x_i^* \le x_i$ E.G.: $x_i^* = x_{i-1}$

$x_i^* = x_i \quad x_i^* = \frac{x_{i-1}+x_i}{2}$

iii) WRITE $S_n = \sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*)$

IF $x_i^* = x_{i-1}$, WE CALL IT L_n

IF $x_i^* = x_i$, WE CALL IT R_n

iv) (TRY TO) COMPUTE $\lim_{n\to\infty} S_n$

SUM RULES

c CONSTANT

$$\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i-b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i b_i \neq \left(\sum_{i=1}^{n} a_i\right)\cdot\left(\sum_{i=1}^{n} b_i\right)$$

$n=2$ $\quad a_1b_1+a_2b_2 \neq (a_1+a_2)(b_1+b_2)$

EX: $\displaystyle\sum_{i=1}^{n} 2i^2+i = 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$

$$\sum_{i=1}^{n} i^2 \neq \left(\sum_{i=1}^{n} i\right)^2$$

EX: $f(x)=x, \quad 1 \leq x \leq 5$

$a=1, \; b=5, \; \Delta x = \frac{5-1}{n} = \frac{4}{n}$

$x_0 = 1, \; x_1 = 1+\frac{4}{n}, \ldots, x_i = 1 + i\cdot\frac{4}{n}$

PICK $x_{i-1} \leq x_i^* \leq x_i$; LET $x_i^* = x_i$

$$R_n = (\Delta x)\sum_{i=1}^{n} f(x_i) = \frac{4}{n}\sum_{i=1}^{n}\left(1+i\cdot\frac{4}{n}\right)$$

$$\sum_{i=1}^{n}\left(1+i\cdot\frac{4}{n}\right) = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} i\cdot\frac{4}{n} = n + \frac{4}{n}\sum_{i=1}^{n} i$$

$$= n + \frac{\cancel{4}^{2}}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}} = n + 2(n+1) = 3n+2$$

$$R_n = \frac{4}{n}\cdot(3n+2) = \frac{12n+8}{n}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{\cancel{n}\left(12+\frac{8}{n}\right)}{\cancel{n}} = \lim_{n\to\infty} 12+\frac{8}{n} = 12$$

$$= \lim_{x\to\infty}\frac{12x+8}{x} \overset{\text{L'HOP}}{=} \lim_{x\to\infty}\frac{12}{1} = 12$$

WHAT IF I PICK $x_i^* = \frac{x_{i-1}+x_i}{2}$?

PROPERTIES: c CONST., f, g INTEGRABLE

i) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

ii) $\int_a^a f(x)\,dx = 0$

iii) $\int_a^b c\,dx = c(b-a)$

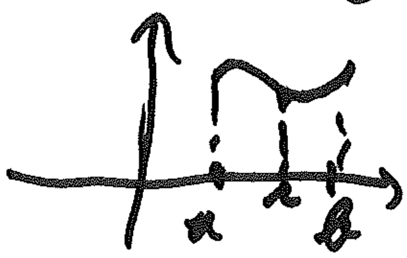
iv) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

v) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

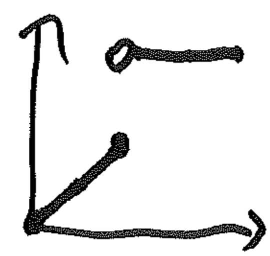
EX: $\int_0^1 3x^2+5\,dx \overset{iv)}{=} \int_0^1 3x^2\,dx + \int_0^1 5\,dx$

$\overset{iii), v)}{=} 3\int_0^1 x^2\,dx + 5\cdot(1-0) \overset{\text{LAST TIME}}{=} 3\cdot\frac{1}{3}+5 = 6$

vi) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



EX: $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2, & 1 < x \le 2 \end{cases}$



$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 2\,dx$

$= \frac{1}{2} + 2 = \frac{5}{2}$

vii) ASSUME $m \leq f(x) \leq M$ FOR m, M CONST.

THEN $\underbrace{m(b-a)}_{\int_a^b m\,dx} \leq \int_a^b f(x)\,dx \leq \underbrace{M(b-a)}_{\int_a^b M\,dx}$

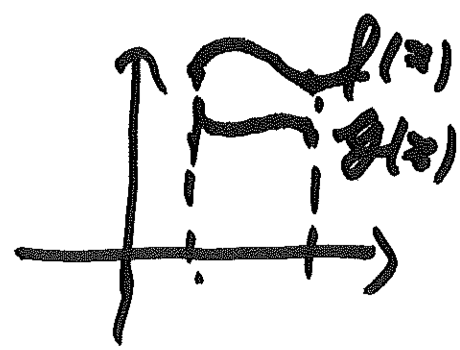
<u>EX</u>: $\int_0^1 \sin(x^2)\,dx$

$0 \leq \sin(x^2) \leq 1$ WHEN $0 \leq x \leq 1$

$0 \leq \int_0^1 \sin(x^2)\,dx \leq 1$

viii) ASSUME $g(x) \leq f(x)$

$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$



<u>EX</u>: $\int_0^1 \sin(x^2)\,dx$

$\sin(x^2) \leq x^2$

$\int_0^1 \sin(x^2)\,dx \leq \int_0^1 x^2\,dx = \frac{1}{3}$