FUNDAMENTAL ~~CALCU~~ THEOREM OF CALCULUS (PART 1)

RECALL: DEFINITE INTEGRAL

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$$\Delta x = \frac{b-a}{n} \qquad x_{i-1} \le x_i^* \le x_i$$

$$S_n = (\Delta x) \sum_{i=1}^{n} f(x_i^*)$$

$$\lim_{n \to \infty} S_n = \int_a^b f(x)\,dx$$

VERY HARD TO COMPUTE!

PICK $f(x)$ CONTINUOUS ON $[a, b]$

DEFINE $F(x) = \int_a^x f(y)\,dy$

x = UPPER LIMIT OF INTEGRATION

y = VARIABLE OF INTEGRATION

y NEVER SHOWS UP IN THE LIMITS OF INTEGRATION

$$F(a) = \int_a^a f(y)\,dy = 0$$

FTC $\underline{I}$: F IS CONTINUOUS ON $[a, b]$

F IS DIFFERENTIABLE ON (a, b)

$$\boxed{F'(x) = f(x)}$$

PICK $a < x < b$, h SMALL

$$F(x+h) - F(x) = \int_a^{x+h} f(y)\,dy - \int_a^x f(y)\,dy$$

$$= \cancel{\int_a^x f(y)\,dy} + \int_x^{x+h} f(y)\,dy - \cancel{\int_a^x f(y)\,dy}$$

$$= \int_x^{x+h} f(y)\,dy$$

BY EVT, $m \le f(y) \le M$ FOR ALL $a \le y \le b$

$$m(x+h-x) \le \int_x^{x+h} f(y)\,dy \le M(x+h-x)$$

$$mh \le \int_x^{x+h} f(y)\,dy \le Mh$$

$$\lim_{h\to 0} mh = m \lim_{h \to 0} h = m \cdot 0 = 0$$

$$\lim_{h\to 0} Mh = 0$$

$$\lim_{h\to 0} \int_x^{x+h} f(y)\,dy = 0$$ BY SQUEEZE THM

$\lim_{h\to 0} F(x+h) - F(x) = 0$, F CONTINUOUS AT x

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(y)\,dy$$

$$\approx \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(x)\,dy=\lim_{h\to 0} f(x)\cdot\frac{1}{h}\cdot$$

$$\cdot\int_x^{x+h} 1\,dy=\lim_{h\to 0} f(x)\cdot\cancel{\frac{1}{h}}\cdot\cancel{h}=f(x)$$

<u>EX 1</u>: $F(x)=\int_2^x \underbrace{\sqrt{t^4+1}}_{\to f(t)}\,dt$

$$F'(x)=f(x)$$

$$F'(x)=\sqrt{x^4+1}$$

<u>EX 2</u>: $G(x)=\int_1^x \sqrt{t^4+1}\,dt$

$$G'(x)=\sqrt{x^4+1}$$

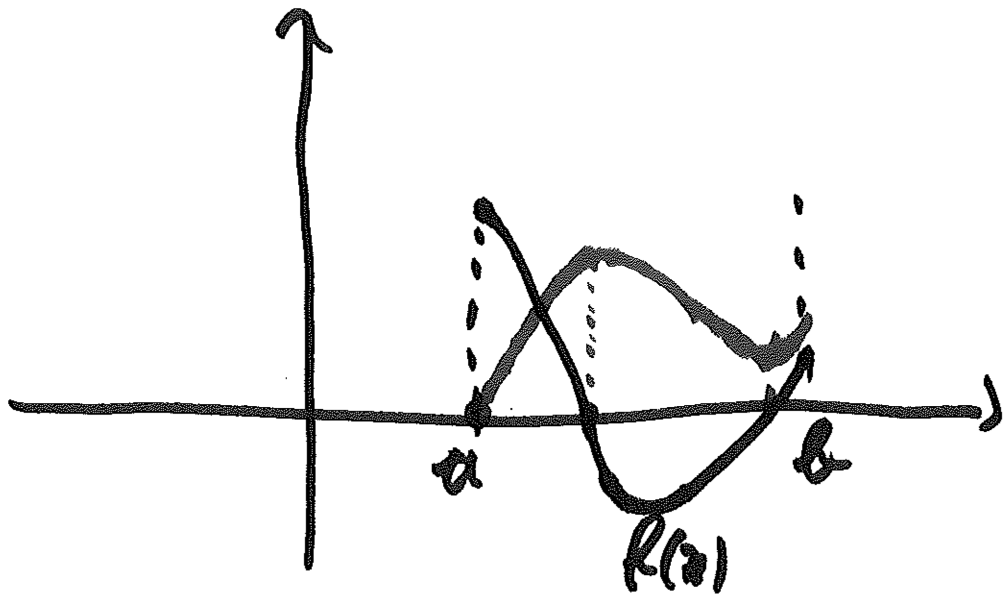
$$G(x)=\int_1^x \sqrt{t^4+1}\,dt=\int_1^2\sqrt{t^4+1}\,dt+\int_2^x\sqrt{t^4+1}\,dt$$

$= F(x)+C$, WHERE $C=\int_1^2\sqrt{t^4+1}\,dt$

GRAPH OF F VS. f

$F'(x) = f(x)$ IF $f(x) > 0$, THEN F INCREASING
$f(x) < 0$, THEN F DECREASING

$F(a) = 0$



SIGN OF f DETERMINES MONOTONICITY OF F

EX 3 : $F(x) = \int_1^{x^2} \sqrt{t^4+1}\,dt$

$F'(x) = ?$

LET $G(x) = \int_1^{x} \sqrt{t^4+1}\,dt$ $\quad G'(x) = \sqrt{x^4+1}$

$F(x) = G(x^2)$ $\quad G'(x^2) = \sqrt{x^8+1}$

$F'(x) = (G(x^2))' = G'(x^2)\cdot(x^2)' = \sqrt{x^8+1}\cdot 2x$

$F'(x) = 2x\sqrt{x^8+1}$

<u>EX 4</u>: $F(x) = \int_{-5}^{3x+7} t^3\,dt$

$$G(x) = \int_{-5}^{x} t^3\,dt \qquad G'(x) = x^3$$

$$G'(3x+7) = (3x+7)^3$$

$$F(x) = G(3x+7)$$

$$F'(x) = G'(3x+7)\cdot(3x+7)' = 3\,(3x+7)^3$$

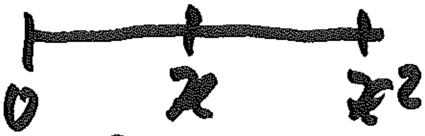
IN GENERAL $F(x) = \int_{a}^{g(x)} f(t)\,dt$

$$F'(x) = f(g(x))\cdot g'(x)$$

<u>EX 5</u>: $\int_{x}^{x^2} \sqrt{t^4+1}\,dt = F(x)$

$F'(x) = ?$

$$\int_{x}^{x^2} \sqrt{t^4+1}\,dt = \int_{0}^{x^2} \sqrt{t^4+1}\,dt - \int_{0}^{x} \sqrt{t^4+1}\,dt$$



$$F'(x) = \left(\int_{0}^{x^2} \sqrt{t^4+1}\,dt - \int_{0}^{x} \sqrt{t^4+1}\,dt\right)'$$

$$= 2x\sqrt{x^8+1} - \sqrt{x^4+1}$$

IN GENERAL, $F(x) = \int_{h(x)}^{g(x)} f(t)\, dt$

$$F'(x) = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$