FUNDAMENTAL THEOREM OF CALCULUS $\underline{\text{II}}$

FTC $\underline{\text{I}}$ $a \le x \le b$, f CONTINUOUS ON $[a,b]$

$$F(x) = \int_a^x f(y)\,dy \text{ ; THEN } F'(x) = f(x)$$

FTC $\underline{\text{II}}$ f CONTINUOUS ON $[a,b]$, THEN

$$\int_a^b f(x)\,dx = F(b) - F(a),$$ WHERE

F IS <u>ANY</u> ANTIDERIVATIVE OF f

<u>PF</u>: BY FTC $\underline{\text{I}}$, $G(x) = \int_a^x f(y)\,dy$ IS

AN ANTIDERIVATIVE

$$G(b) - G(a) = \int_a^b f(y)\,dy - \overbrace{\int_a^a f(y)\,dy}^{0}$$

IF $F(x)$ <u>ANY</u> ANTIDERIVATIVE, THEN

$F(x) = G(x) + C$ FOR SOME C

$$F(b) - F(a) = (G(b) + C) - (G(a) + C)$$

$$= G(b) + \cancel{C} - G(a) - \cancel{C} = \int_a^b f(y)\,dy$$

BLUEPRINT FOR COMPUTING $\int_a^b f(x)\,dx$

<u>STEP 1</u>: FIND AN ANTIDERIVATIVE F OF f

<u>STEP 2</u>: $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>EX 1</u>: $\int_2^5 4\,dx$

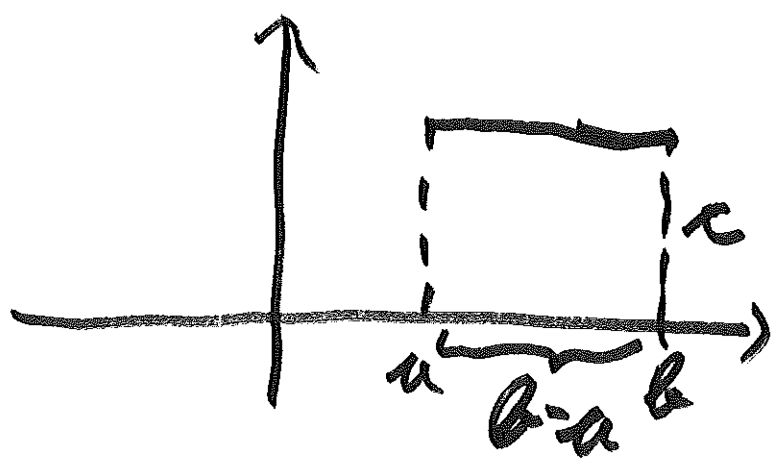
$f(x) = 4 \quad F(x) = 4x$ AN ANTIDERIVATIVE

$$\int_2^5 4\,dx = F(5) - F(2) = 4\cdot 5 - 4\cdot 2 = 12$$

IN GENERAL, $\int_a^b c\,dx$

$f(x) = c,\ F(x) = cx$

$$\int_a^b c\,dx = F(b) - F(a) = cb - ca = c(b-a)$$



EX 2: $\int_0^1 x^2\,dx$

$f(x) = x^2 \qquad F(x) = \frac{1}{3}x^3$

$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$

EX 3: $\int_0^{\frac{\pi}{2}} \cos x\,dx$

RIEMANN SUMS

$\Delta x = \frac{\frac{\pi}{2}}{n} \qquad x_i = 0 + i\,\Delta x = i \cdot \frac{\pi}{2n}$

$x_{i-1} \le x_i^* \le x_i$

$L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$

$= \frac{\pi}{2n} \sum_{i=1}^{n} \cos\left(\frac{\pi}{2n}(i-1)\right)$ VERY MESSY

FTC II

$f(x) = \cos x \qquad F(x) = \sin x$

$\int_0^{\frac{\pi}{2}} \cos x\,dx = \sin\frac{\pi}{2} - \sin 0 = 1$

EX 4: $\int_1^2 \frac{1}{x^2}\,dx$

$f(x) = \frac{1}{x^2} = x^{-2} \qquad F(x) = -x^{-1}$

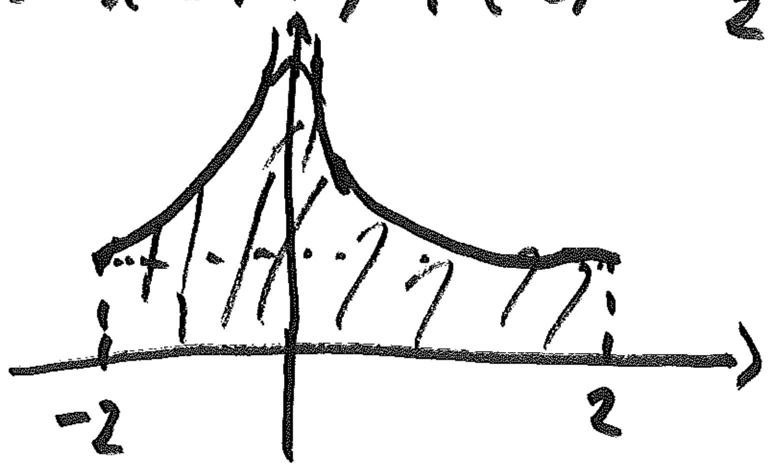
$\int_1^2 \frac{1}{x^2}\,dx = F(2) - F(1) = -\frac{1}{2} - (-1) = -\frac{1}{2} + 1 = \frac{1}{2}$

$\int_{-2}^2 \frac{1}{x^2}\,dx$ DNE

$f(x) = x^{-2}, \; F(x) = -x^{-1} \qquad F(2) = -\frac{1}{2}$

$F(-2) = \frac{1}{2}$

$\int_{-2}^2 \frac{1}{x^2}\,dx = F(2) - F(-2) = -\frac{1}{2} - \left(\frac{1}{2}\right) = -1$



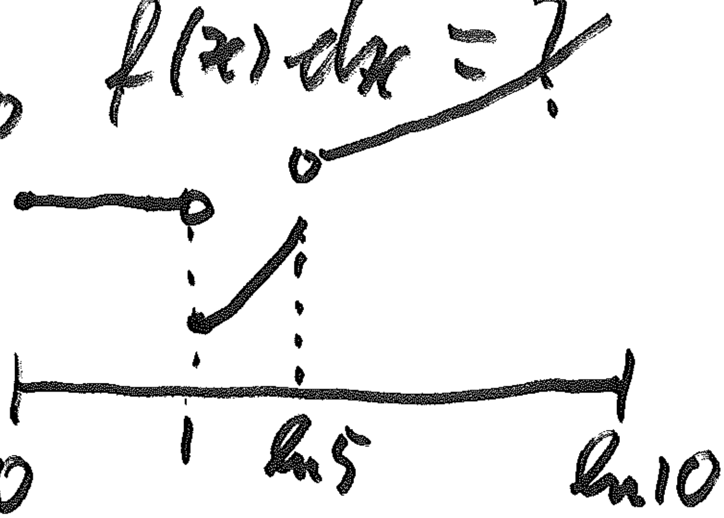
PROBLEM: $\frac{1}{x^2}$ IS NOT CONTINUOUS ON $[-2, 2]$

IN FACT, IT HAS INFINITE TYPE OF DISCONTINUITY AT $x = 0$

PIECEWISE CONTINUOUS FUNCTION

<u>EX</u> : $f(x) = \begin{cases} 5, & 0 \le x < 1 \\ x^2, & 1 \le x < \ln 5 \\ e^x, & \ln 5 \le x < \ln 10 \end{cases}$

$\int_0^{\ln 10} f(x)\,dx = ?$



$$\int_0^{\ln 10} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{\ln 5} f(x)\,dx + \int_{\ln 5}^{\ln 10} f(x)\,dx$$

$$= \int_0^1 5\,dx + \int_1^{\ln 5} x^2\,dx + \int_{\ln 5}^{\ln 10} \cancel{f(x)}\, e^x\,dx$$

$$= 5 + \frac{(\ln 5)^3}{3} - \frac{1^3}{3} + \underbrace{e^{\ln 10}}_{10} - \underbrace{e^{\ln 5}}_{5}$$

$$= 10 + \frac{(\ln 5)^3 - 1}{3}$$

FTC I VS. II

"DERIVATIVES AND INTEGRALS ARE INVERSE TO EACH OTHER"

FTC I : $\frac{d}{dx}\int_a^x f(y)\,dy = f(x)$

FTC II : $\int_a^b \frac{d}{dx}F(x)\,dx = F(b) - F(a)$

USE PROPERTIES OF THE INTEGRAL

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

EX: $\int_0^1 \frac{3}{1+x^2} + 7\cdot 2^x\,dx$

$$= \int_0^1 \frac{3}{1+x^2}\,dx + \int_0^1 7\cdot 2^x\,dx \qquad \frac{1}{1+x^2} \rightsquigarrow \tan^{-1}x$$

$$= 3\int_0^1 \frac{1}{1+x^2}\,dx + 7\int_0^1 2^x\,dx \qquad 2^x \rightsquigarrow \frac{1}{\ln 2}\cdot 2^x$$

$$= 3\cdot(\tan^{-1}1 - \tan^{-1}0) + 7\cdot\left(\frac{2^1}{\ln 2} - \frac{2^0}{\ln 2}\right)$$

$= 3 \cdot \left(\frac{\pi}{4} - 0\right) + \frac{7}{\ln 2} = \frac{3\pi}{4} + \frac{7}{\ln 2}$

$\int_0^1 \frac{x}{1+x^2}\,dx$

$\ln(1+x^2)' = \frac{(1+x^2)'}{1+x^2}$

$= \frac{2x}{1+x^2}$

$= \frac{1}{2}\ln(1+1^2) - \frac{1}{2}\ln(1+0^2)$

$= \frac{1}{2}\ln 2$

$\frac{2x}{1+x^2} \rightsquigarrow \ln(1+x^2)$

$\frac{x}{1+x^2} \rightsquigarrow \frac{1}{2}\ln(1+x^2)$