HIGHER-ORDER APPROXIMATIONS (TAYLOR POLYNOMIALS)

i) ESTIMATE $\sqrt{2}$

$f(x) = \sqrt{1+x} \qquad f(1) \approx ?$

USE LINEARIZATION AROUND 0

$L(x) = f(0) + f'(0)x$

$f(0) = \sqrt{1+0} = 1$

$f'(x) = \dfrac{1}{2\sqrt{1+x}} \qquad f'(0) = \dfrac{1}{2}$

$L(x) = 1 + \frac{1}{2}x$

$f(1) \approx L(1) = \frac{3}{2} \qquad \sqrt{2} \approx 1.5$

TRY A QUADRATIC APPROXIMATION

LOOK AT $T_2(x) = f(0) + f'(0)x + \dfrac{f''(0)x^2}{2}$

$f(1) \approx T_2(1) = 1 + \frac{1}{2} - \frac{1}{8} = 1.375$

$f''(x) = -\frac{1}{4} \cdot \dfrac{1}{(\sqrt{1+x})^3} \qquad f''(0) = -\frac{1}{4}$

$T_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$f(1) \approx T_3(1) = 1.4375$$

$T_n(x)$ = TAYLOR POLYNOMIALS

$T_1(x)$ = LINEARIZATION

$T_2(x)$ = QUADRATIC POLYNOMIAL

$\vdots$

$T_n(x) = n^{TH}$ ORDER POLYNOMIAL

HOW DO WE COMPUTE $T_n(x)$ FOR GENERAL n?

PICK $n = 2$, a NUMBER

$$f(x) \approx a_0 + a_1(x-a) + a_2(x-a)^2 = g(x)$$

WHEN x IS CLOSE TO a

$f(x)$ CLOSE TO $f(a)$

$a_0 + a_1(x-a) + a_2(x-a)^2$ CLOSE TO a_0

SAY $f'(x) \approx g'(x) = a_1 + 2a_2(x-a)$

$f'(x) \approx f'(a) \quad g'(x) \approx a_1$

SO $a_1 = f'(a)$

SAY $f''(x) \approx g''(x) = 2a_2$

$f''(x) \approx f''(a) \quad g''(x) \approx 2a_2$

$a_2 = \frac{f''(a)}{2}$

FOR $T_3(x) \quad a_3(x-a)^3$

$f'''(x) \approx \left(a_3(x-a)^3\right)''' = 2\cdot 3\cdot a_3$

$a_3 = \frac{f'''(a)}{6}$

$f''''(x) \approx \left(a_4(x-a)^4\right)'''' = 2\cdot 3\cdot 4\, a_4$

$a_4 = \frac{f''''(a)}{4!} \qquad n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$

FOR n^{TH} COEFF.: $a_n = \frac{f^{(n)}(a)}{n!}$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>EX</u>: $f(x) = \sin x$ NEAR $x=0$

WHAT IS $T_6(x)$?

$f(0) = 0 \quad f'(x) = \cos x, \; f'(0) = 1$

$f''(x) = -\sin x, \; f''(0) = 0$

$f'''(x) = -\cos x, \; f'''(0) = -1$

$f^{(4)}(x) = +\sin x, \; f^{(4)}(0) = 0$

$0, 1, 0, -1, 0, 1, 0, -1, \dots$

$$T_6(x) = 0 + 1\cdot x + \frac{0}{2}x^2 + \frac{(-1)}{6}x^3 + \frac{0}{24}x^4 + \frac{1}{120}x^5 + \frac{0}{6!}x^6 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\sin\frac{\pi}{100} \approx \frac{\pi}{100} - \frac{1}{6}\left(\frac{\pi}{100}\right)^3 + \frac{1}{120}\left(\frac{\pi}{100}\right)^5$$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

LAST TIME: APPROX. $\sqrt{24}$

APPROX. #1 $\sqrt{24} \approx \sqrt{25} = 5$

APPROX. #2 $\sqrt{24} \approx 5 - 0.1 = 4.9$

$f(x) = \sqrt{x}$, LINEAR APPROX. NEAR $a = 25$

APPROX. #3 COMPUTE $T_2(x)$ NEAR $a = 25$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad f''(x) = -\frac{1}{4x^{3/2}} \qquad f'(25) = \frac{1}{10}$$

$$f''(25) = -\frac{1}{4 \cdot 25^{3/2}} = -\frac{1}{600}$$

$$T_2(x) = 5 + \frac{1}{10} \cdot (x - 25) - \frac{1}{1200}(x - 25)^2$$

$$\sqrt{24} \approx T_2(24) = 5 - \frac{1}{10} - \frac{1}{1200} = 4.8983$$

EX: $f(x)=e^x$ WHEN $a=0$

$f'(x)=e^x \quad f^{(n)}(x)=e^x \quad f^{(n)}(0)=1$

$T_n(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots+\frac{1}{n!}x^n$

LET $x=1$

$e\approx 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\dots$

EX: APPROXIMATE $\int_0^1 e^{x^2}dx$

$e^{x^2}\approx T_n(x^2)=1+x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\dots+\frac{1}{n!}x^{2n}$

$n=3$

$e^{x^2}\approx 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}$

$\int_0^1 e^{x^2}dx\approx\int_0^1 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}\,dx$

$=x+\frac{x^3}{3}+\frac{x^5}{10}+\frac{x^7}{42}\Big|_0^1$

$=1+\frac{1}{3}+\frac{1}{10}+\frac{1}{42}$