LIMIT LAWS

$$\lim_{x \to a} f(x) \begin{cases} \to \text{DNE} \\ \to L, \quad -\infty < L < +\infty \\ \to -\infty \\ \to +\infty \end{cases}$$

L IS A <u>NUMBER</u>

EX: $\displaystyle \lim_{x \to 0} \frac{\sqrt{x^2+1}}{x^3+2x-5}$

LIMIT LAWS

1) $f(x) = C$, C CONSTANT

$$\lim_{x \to a} f(x) = C$$

2) $f(x) = x$

$$\lim_{x \to a} f(x) = a$$

3) $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ EXIST, FINITE

THEN $\displaystyle \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$\underline{EX}$: $\lim\limits_{x \to 3} (x+5) = \lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 5$

$$= 3 + 5 = 8$$

4) $\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

5) C CONSTANT

$$\lim_{x \to a} [C f(x)] = C \lim_{x \to a} f(x)$$

$\underline{EX}$: $\lim\limits_{x \to 3} (2x - 7) \overset{4}{=} \lim\limits_{x \to 3} 2x - \lim\limits_{x \to 3} 7$

$$\overset{5}{=} 2 \lim_{x \to 3} x - \lim_{x \to 3} 7 \overset{2}{=} 2 \cdot 3 - \lim_{x \to 3} 7$$

$$\overset{1}{=} 2 \cdot 3 - 7 = -1$$

6) $\lim\limits_{x \to a} (f(x) \cdot g(x)) = [\lim\limits_{x \to a} f(x)] \cdot [\lim\limits_{x \to a} g(x)]$

$\underline{EX}$: $\lim\limits_{x \to 7} x^2 \overset{6}{=} \lim\limits_{x \to 7} x \cdot \lim\limits_{x \to 7} x = 7 \cdot 7 = 49$

IN PARTICULAR

$$\lim_{x \to a} x^n = a^n$$

$$\lim_{x \to a} (f(x))^n = \left[\lim_{x \to a} f(x)\right]^n$$

<u>EX</u>: $\lim_{x \to 1} (2x+1)^5 = \left[\lim_{x \to 1} (2x+1)\right]^5$

$$= \left[2 \lim_{x \to 1} 1 + \lim_{x \to 1} 1\right]^5 = (2 \cdot 1 + 1)^5 = 3^5 = 243$$

7) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, IF $\lim_{x \to a} g(x) \neq 0$

8) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (WHEN WELL-DEFINED)

DIRECT SUBSTITUTION:

IF f IS A POLYNOMIAL/RATIONAL FUNCTION AND a IS IN THE DOMAIN OF f, THEN

$$\lim_{x \to a} f(x) = f(a)$$

WHAT IF $f(x) = \dfrac{P(x)}{Q(x)}$

$\lim_{x \to a} f(x)$ IF $Q(a) = 0$

IF $P(a) = 0$ "$\frac{0}{0}$" CASE

ELIMINATE COMMON FACTORS OF P AND Q

$$\lim_{x \to a} \frac{x+\cancel{a}}{x+\cancel{a}} = \lim_{x \to a} 1 = 1$$

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x \to 2} (x+2)$$

$$= 4$$

$x=2$: $\dfrac{2^2-4}{2-2} = \dfrac{0}{0}$

$x^2 - 4 = (x-2)(x+2)$

<u>OR</u> : $x-2=h$, $x=2+h$

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{h\to 0}\frac{(2+h)^2-4}{h}$$

$$=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}=\lim_{h\to 0}\frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$=\lim_{h\to 0}(4+h)=4$$

<u>EX</u> : $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

$x=0$: $\dfrac{\sqrt{0+1}-1}{0}=$ "$\dfrac{0}{0}$"

$\sqrt{a}-\sqrt{b}$ CONJUGATE $\sqrt{a}+\sqrt{b}$

$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$

$$\frac{\sqrt{x+1}-1}{x}=\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}=\frac{x\cancel{+1}-\cancel{1}}{x(\sqrt{x+1}+1)}$$

$$=\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}=\frac{1}{\sqrt{x+1}+1}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{2}$$

WHAT IF $P(a) \neq 0$ AND $Q(a) = 0$?

THEN THE LIMIT MIGHT BE $+\infty$, $-\infty$, DNE

EX: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ DNE

$$\lim_{x \to 2^+} \frac{x^2+4}{x-2} = +\infty$$

$$\lim_{x \to 2^-} \frac{x^2+4}{x-2} = -\infty$$

$$\lim_{x \to 2} \frac{x^2+4}{(x-2)^2} = +\infty$$

$$f(x) = \begin{cases} \sqrt{x-2}, & x > 2 \\ 2x-4, & x < 2 \end{cases}$$

$\lim_{x \to 2} f(x) = 0$

$\lim_{x \to 2^+} f(x) = 0$

$\lim_{x \to 2^-} f(x) = 0$

SQUEEZE THM

IF $f(x) \leq g(x) \leq h(x)$ NEAR a (EXCEPT MAYBE AT a), AND

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

THEN $\lim_{x \to a} g(x) = L$

EX: $-x^2 \leq g(x) \leq x^2$

$$\lim_{x \to 0} g(x) = 0$$