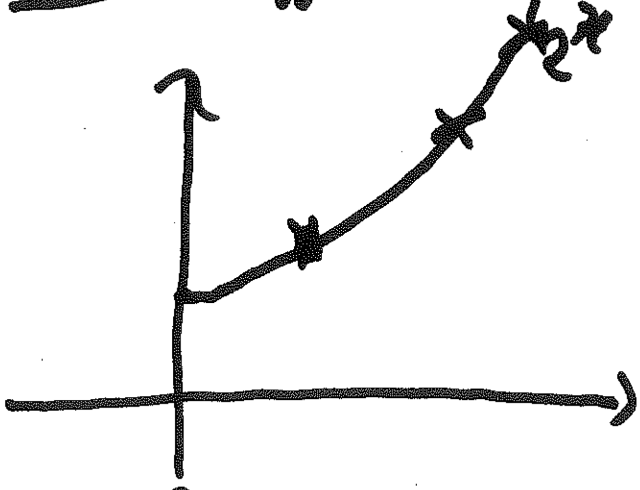
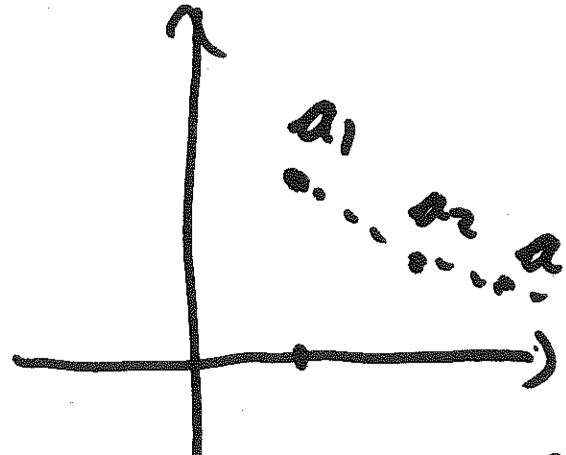
SEQUENCES BY RECURSION

RECALL: $\{a_n\}$ IS A SEQUENCE IF THERE IS $f: \mathbb{N} = \{1, 2, 3, \ldots\} \to \mathbb{R}$ SO THAT $f(n) = a_n$

<u>EX</u>: $a_n = 2^n$ $\quad a_n = \frac{n+1}{n}$





$\{a_n\}$ <u>converges</u> if there is L so that $\lim_{n\to\infty} a_n = L$; for all $\varepsilon > 0$, $|a_n - L| < \varepsilon$ if n is large enough

$\{a_n\}$ diverges if it does not converge

CONVERGENCE CRITERION:

If $\lim_{x\to\infty} f(x) = L$, and $a_n = f(n)$, then
$$\lim_{n\to\infty} a_n = L$$

IF $\{a_n\}$ is monotone and bounded then $\{a_n\}$ convergent.

RECURSIVE SEQUENCES

a_n depends on $a_{n-1}, a_{n-2}, \dots, a_1$

NO EXPLICIT FORMULA FOR a_n

<u>EX 1</u>: $a_1 = 1$, $a_2 = 2a_1$, $a_3 = 2a_2, \dots$

$$a_{n+1} = 2a_n$$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2$

$a_3 = 2 \cdot a_2 = 2 \cdot 2 = 4 = 2^2$

$a_4 = 2 \cdot a_3 = 2 \cdot 4 = 8 = 2^3$

$\vdots$

$a_n = 2^{n-1}$

$a_{n+1} = 2 \cdot a_n = 2 \cdot 2^{n-1} = 2^n$

EX 2: $F_1 = 1,\ F_2 = 1$

$$F_{n+1} = F_n + F_{n-1}$$

$$F_3 = F_2 + F_1 = 1+1 = 2$$

$$F_4 = F_3 + F_2 = 2+1 = 3$$

$$F_5 = F_4 + F_3 = 3+2 = 5$$

$$F_6 = 8,\ F_7 = 13,\ F_8 = 21,\ F_9 = 34$$

GUESS: $F_n \approx r^n$

$$r^{n+1} \text{"="} r^n + r^{n-1}$$

$$r^2 = r+1 \quad ,\ r^2 - r - 1 = 0$$

$$r = \frac{1+\sqrt{5}}{2} \quad \textit{golden ratio}$$

$$F_n \approx \left(\frac{1+\sqrt{5}}{2}\right)^n \to +\infty \quad \text{DIVERGENT}$$

EX 3: $a_1 = 0$

$$a_{n+1} = \frac{1}{2} a_n + 1$$

Q1: Find a_n

Q2: $\lim_{n \to \infty} a_n = ?$

$$a_2 = \frac{1}{2} \cdot 0 + 1 = 1$$

$$a_3 = \frac{1}{2} \cdot 1 + 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$a_4 = \frac{1}{2} \cdot \frac{3}{2} + 1 = \frac{3}{4} + 1 = \frac{7}{4} = \frac{2^3 - 1}{2^2}$$

$$a_5 = \frac{1}{2} \cdot \frac{7}{4} + 1 = \frac{7}{8} + 1 = \frac{15}{8} = \frac{2^4 - 1}{2^3}$$

Guess: $a_n = \dfrac{2^{n-1} - 1}{2^{n-2}}$

$$a_{n+1} = \frac{1}{2} \cdot \frac{2^{n-1} - 1}{2^{n-2}} + 1 = \frac{2^{n-1} - 1}{2^{n-1}} + 1$$

$$= \frac{2^{n-1} - 1 + 2^{n-1}}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$$

✓

$$\lim_{n\to\infty} \frac{2^n-1}{2^{n-1}} = \lim_{n\to\infty} \frac{2^n}{2^{n-1}} - \frac{1}{2^{n-1}}$$

$$= \lim_{n\to\infty} 2 - \frac{1}{2^{n-1}} = 2$$

LOOK AT $a_{n+1} = f(a_n)$, f continuous

$$\lim_{n\to+\infty} a_n = ?$$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} f(a_n)$$

$$L = f(\lim_{n\to+\infty} a_n) = f(L)$$

L must be a fixed point of f

<u>EX 3</u>: ASSUME $\lim_{n\to+\infty} a_n = L$

PASS TO LIMIT

$L = \frac{1}{2}L + 1$, $\frac{1}{2}L = 1$, $L = 2$

EX 4: COMPUTE

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}} = 2$$

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2+\sqrt{2}}$$

$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\vdots$$

$$\lim_{n\to\infty} a_n = ?$$

$$a_{n+1} = \sqrt{2+a_n}$$

PASS TO LIMIT

$$L = \sqrt{2+L}$$

$$L^2 = 2+L \quad L^2 - L - 2 = 0$$

$$L = 2$$

$$a_{n+1} = 1 + \frac{1}{a_n}, \quad a_1 = 1$$

PASS TO LIMIT:

$$L = 1 + \frac{1}{L}, \quad L^2 = L + 1$$

$$L^2 - L - 1 = 0, \quad L = \frac{1+\sqrt{5}}{2}$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{1+\sqrt{5}}{2}$$

$$a_2 = 1 + \frac{1}{1} = \frac{2}{1}$$

$$a_3 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\vdots$$

$$a_n = \frac{F_{n+1}}{F_n} \longrightarrow \frac{1+\sqrt{5}}{2}$$