SERIES

$\{a_n\}$ SEQUENCE

$$\sum a_n = \lim_{n\to\infty} s_n, \quad s_n = a_1 + a_2 + \dots + a_n$$

IF THE LIMIT EXISTS, FINITE, THEN THE SERIES CONVERGES OTHERWISE, IT DIVERGES

GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

$$= \begin{cases} \dfrac{a}{1-r}, & \text{IF } |r| < 1 \\ \text{diverges}, & \text{IF } |r| \geq 1 \end{cases}$$

<u>EX</u>: $\sum_{n=1}^{\infty} \frac{(-2)^{2n+1}}{5^n} = \sum_{n=1}^{\infty} a r^{n-1}$

$$(-2)^{2n+1} = (-2)^{2n} \cdot (-2) = 4^n \cdot (-2)$$

$$= 4^{n-1} \cdot 4 \cdot (-2) = -8 \cdot 4^{n-1}$$

$$5^n = 5^{n-1} \cdot 5$$

$$\frac{(-2)^{2n+1}}{5^n} = \frac{-8 \cdot 4^{n-1}}{5 \cdot 5^{n-1}} = -\frac{8}{5} \cdot \left(\frac{4}{5}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{2n+1}}{5^n} = \sum_{n=1}^{\infty} \underbrace{\left(-\frac{8}{5}\right)}_{a} \cdot \underbrace{\left(\frac{4}{5}\right)}_{r<1}^{n-1}$$

$$= \frac{a}{1-r} = \frac{-\frac{8}{5}}{1-\frac{4}{5}} = \frac{-\frac{8}{5}}{\frac{1}{5}} = -8$$

FUN EXAMPLE

$$a_n = \frac{1}{n}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

$$s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \qquad s_n < s_{n+1}$$

$$s_{n+1} = s_n + \frac{1}{n+1}$$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \dots + \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \dots + \frac{1}{2^{n+1}}\right) + \dots$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}} + \dots$$

infinitely many $\frac{1}{2}$ DIVERGES

Q: HOW DO WE KNOW IF A SERIES CONVERGES?

SUPPOSE $\sum a_n = L$, L FINITE

$s_n = a_1 + a_2 + \ldots + a_n$

$\lim_{n \to \infty} s_n = L$

$$a_n = (a_1 + a_2 + \ldots + a_n) - (a_1 + \ldots + a_{n-1})$$
$$= s_n - s_{n-1}$$

$\lim_{n \to \infty} s_{n-1} = L$

$$\lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1}$$
$$= L - L = 0$$

$\lim_{n \to \infty} a_n = 0$

THM: IF $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$

DIVERGENCE TEST

If $\lim\limits_{n\to\infty} a_n \neq 0$, or DNE, then the series diverges

EX 1: $a_n = 1 \qquad \sum\limits_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} a_n = 1 \neq 0, \quad \text{DIVERGES}$$

EX 2: $a_n = \dfrac{n^3 - n + 2}{5n^3+1} \qquad \sum\limits_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} a_n = \frac{\infty}{\infty} \qquad \text{DIVERGES}$$

$$\lim_{n\to\infty} \frac{n^3-n+2}{5n^3+1} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1-\frac{1}{n^2}+\frac{2}{n^3}\right)}{\cancel{n^3}\left(5+\frac{1}{n^3}\right)}$$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n^2}+\frac{2}{n^3}}{5+\frac{1}{n^3}} = \frac{1}{5} \neq 0$$

$$\lim_{x\to\infty} \frac{x^3-x+2}{5x^3+1} \quad \text{L'HOPITAL} \quad \frac{1}{5}$$

EX 3: GEOMETRIC SERIES

$a_n = r^n$

$\lim_{n \to \infty} r^n = 0$ IF AND ONLY IF $|r| < 1$

DIVERGENCE THM: IF $|r| \geq 1$, THEN THE SERIES DIVERGES

PROPERTIES:

$\{a_n\}, \{b_n\}$ SO THAT $\sum a_n, \sum b_n$ CONVERGE. LET c BE A CONSTANT. THEN:

i) $\sum c a_n = c \sum a_n$

ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$

$\underline{EX}$: $\displaystyle\sum_{n=1}^{\infty}\left[\left(\frac{5}{6}\right)^{n+1}-\frac{3}{n(n+1)}\right]=\frac{25}{6}-3=\frac{7}{6}$

$a_n=\left(\frac{5}{6}\right)^{n+1}=\left(\frac{5}{6}\right)^{n-1}\cdot\left(\frac{5}{6}\right)^2=\underbrace{\frac{25}{36}}_{a}\cdot\underbrace{\left(\frac{5}{6}\right)}_{r}^{n-1}$

$b_n=\frac{3}{n(n+1)}$

$|r|<1$

$\displaystyle\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=$

$\displaystyle\boxed{\sum b_n=\sum 3\cdot\frac{1}{n(n+1)}=3\sum\frac{1}{n(n+1)}\stackrel{LT}{=}3\cdot 1=3}$

$=\frac{25}{36}\cdot\frac{1}{1-\frac{5}{6}}=\frac{25}{36}\cdot 6=\frac{25}{6}$

THE FIRST TERMS OF A SERIES DO NOT AFFECT CONV./DIV. !