SERIES

$\{a_n\}$ sequence

$$s_N = a_1 + \dots + a_N = \sum_{n=1}^{N} a_n$$

$\lim_{N \to \infty} s_N = L$ finite? $\rightarrow$ YES, CONVERGENT; $\rightarrow$ NO, DIVERGENT

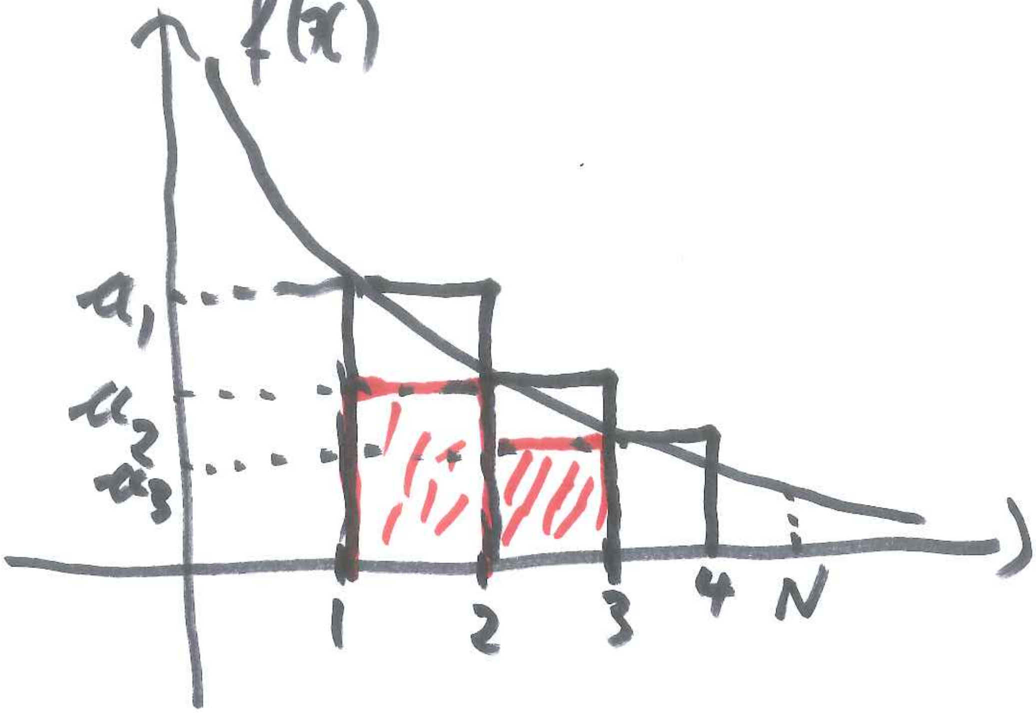
<u>Q1</u>: DOES A SERIES CONVERGE?

<u>Q2</u>: IF YES, WHAT IS ITS VALUE?

ASSUME $a_n = f(n)$

<u>EX</u>: $a_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$

$a_n = \frac{1}{2^n}$, $f(x) = \frac{1}{2^x}$



IMPROPER INTEGRAL OF f

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

series $\sum a_n$	$\int_1^\infty f(x)\,dx$
a_n	$f(n)$
s_N	$\int_1^N f(x)\,dx$
$s = \lim_{N\to\infty} s_N$	$\lim_{N\to\infty} \int_1^N f(x)\,dx = \int_1^\infty f(x)\,dx$

$$s_{N-1} = a_1 + a_2 + \dots + a_{N-1} \geq \int_1^N f(x)\,dx$$

$$a_2 + \dots + a_{N-1} \leq \int_1^N f(x)\,dx$$

INTEGRAL TEST

$a_n = f(n)$, f continuous, positive, decreasing on $[1,\infty)$. Then:

i) If $\int_1^\infty f(x)\,dx$ converges, then $\sum a_n$ converges

ii) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges

CAREFUL: $\sum a_n \neq \int_1^\infty f(x)\,dx$

$a_n = \frac{1}{n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$f(x) = \frac{1}{x}$ $\quad \int_1^\infty f(x)\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$

$= \lim_{R\to\infty} \ln x \Big|_1^R = \lim_{R\to\infty} (\ln R - \ln 1) = +\infty$

CONVERGES

EX: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad a_n = f(n),$

$$f(x) = \frac{1}{x^2}$$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R x^{-2}\,dx$$

$$= \lim_{R\to\infty} \left(-x^{-1}\right)\Big|_1^R = \lim_{R\to\infty} \left(1 - R^{-1}\right) = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$$

EX: Pick p, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\to$ CONV. $p > 1$; $\to$ DIV. $p \le 1$

p-series

EX: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\quad f(x) = \dfrac{1}{x \ln x}$

$$\int_2^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x \ln x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{u}\,du =$$

$$= \lim_{R\to\infty} \ln u \Big|_{\ln 2}^{\ln R} = \lim_{R\to\infty} \left[\ln(\ln R) - \ln \ln 2\right]$$

$$= +\infty$$

Remainder estimate

ASSUME $\lim_{N\to\infty} s_N = s$; look at

$$R_N = s - s_N = a_{N+1} + a_{N+2} + \ldots$$

Q: HOW LARGE IS R_N?

THM: $\int_{N+1}^{\infty} f(x)\,dx \le R_N \le \int_N^{\infty} f(x)\,dx$

EX: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$

$$s_5 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \approx 1.4636$$

HOW LARGE IS R_5?

$$\int_6^{\infty} \frac{1}{x^2}\,dx \le R_5 \le \int_5^{\infty} \frac{1}{x^2}\,dx$$

$$\frac{1}{6} \le R_5 \le \frac{1}{5}$$

$$s_5 + \frac{1}{6} \le \sum_{1}^{\infty} \frac{1}{n^2} \le s_5 + \frac{1}{5}$$

1.6302 1.6636