Secret code: ____________ Page _____ of _____

TRIGONOMETRIC SUBSTITUTION

$\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2+x^2}$

EX 1: $\int \frac{1}{\sqrt{1-x^2}}\,dx$

$x = \sin u$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$

$1-x^2 = 1-\sin^2 u = \cos^2 u$

$\sqrt{\cos^2 u} = |\cos u| = \cos u$

$dx = \cos u\, du$

$\int \frac{1}{\cancel{\cos u}} \cdot \cancel{\cos u}\, du = \int 1\, du = u + C$

$u = \arcsin x \qquad = \arcsin x + C$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin \frac{x}{a} + C$

$x = a \sin u$, $a^2 - x^2 = a^2 \cos^2 u$

EX 2: $\int \sqrt{1-x^2}\,dx$

$x = \sin u$, $u = \arcsin x$

$dx = \cos u\,du$

$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$

$\int (\cos u)\cdot \cos u\,du = \int \cos^2 u\,du$

$= \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C$

$= \frac{1}{2}\arcsin x + \frac{x\sqrt{1-x^2}}{2}$

$\frac{\sin 2u}{4} = \frac{2\sin u \cos u}{4} = \frac{\cancel{2}\,x\sqrt{1-x^2}}{\cancel{4}\,2}$

$\int \sqrt{a^2-x^2}\,dx$ $\qquad x = a\sin u$

Ex 3: $\int x\sqrt{1-x^2}\,dx$

$1-x^2=u$

$-2x\,dx=du,\ x\,dx=-\frac{1}{2}du$

$=\int\sqrt{u}\cdot\left(-\frac{1}{2}du\right)$

$=-\frac{1}{2}\int u^{\frac{1}{2}}du=\left(-\frac{1}{2}\right)\cdot\frac{2}{3}u^{\frac{3}{2}}$

$=-\frac{1}{3}u^{\frac{3}{2}}=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$

IN GENERAL:

$\sqrt{a^2-x^2}$ shows up, can try $x=a\sin u$

Secret code: ________ Page ____ of ____

EX 4: $\int \frac{1}{\sqrt{1+x^2}}\,dx$

$x = \tan u$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$

$\sqrt{1+x^2} = \sqrt{1+\tan^2 u} = \sqrt{\sec^2 u}$

$= \sec u = \frac{1}{\cos u}$

$dx = \sec^2 u\,du$

$\int \frac{1}{\sec u} \cdot \sec^2 u\,du = \int \sec u\,du$

$= \ln|\sec u + \tan u| + C$

$= \ln|\sqrt{1+x^2} + x| + C$

$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ldots$

$x = a\tan u$

EX 5: $\int x\sqrt{1+x^2}\,dx$

$$1+x^2=u$$
$$du=2x\,dx$$
$$x\,dx=\frac{1}{2}du$$

$$\int \sqrt{u}\cdot\frac{1}{2}du=\frac{1}{2}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}+C$$
$$=\frac{1}{3}u^{\frac{3}{2}}+C=\frac{1}{3}(1+x^2)^{\frac{3}{2}}+C$$

EX 6: $\int \frac{1}{\sqrt{x^2-1}}dx$ $\quad \boxed{x>1}$ or $x<-1$

$$x=\sec u$$
$$\sqrt{x^2-1}=\sqrt{\sec^2 u-1}=\sqrt{\tan^2 u}=\tan u$$
$$dx=(\sec u)'du=\sec u\tan u\,du$$

$$\int \frac{1}{\tan u} \cdot \sec u \tan u \, du$$

$$= \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$= \ln|x + \sqrt{x^2 - 1}| + C$$

MORAL:

$\sqrt{a^2 - x^2}$ $\quad x = a \sin u$

$\sqrt{a^2 + x^2}$ $\quad x = a \tan u$

$\sqrt{x^2 - a^2}$ $\quad x = a \sec u$

CAVEAT: SOMETIMES EASIER