RATIONAL FUNCTIONS

$R(x) = \frac{P(x)}{Q(x)}$ P, Q polynomials

EX: $R(x) = \frac{x^3 - x + 5}{7x^2 - 2x - 7}$

If $Q(x) \equiv 1$ ✓

EX: $\int x^2 + 1 \, dx = \frac{x^3}{3} + x + C$

$Q(x) = x$, $P(x) \equiv 1$

$\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{2x+3} dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \ln|u| + C$

$= \frac{1}{2} \ln|2x+3| + C$

$2x + 3 = u$

$2 dx = du$, $dx = \frac{1}{2} du$

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$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$u = ax+b$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \arctan\frac{x}{a} + C$$

$$\int \frac{1}{x^2-4x+8}\,dx = \int \frac{1}{(x-2)^2+2^2}\,dx \overset{u=x-2}{=} \int \frac{1}{u^2+2^2}\,du$$

$x^2-4x+8 = x^2-2\cdot x\cdot 2+8 = x^2-2\cdot x\cdot 2+4+4$

$\boxed{(a-b)^2 = a^2-2ab+b^2}$

$= (x-2)^2+2^2$

$$= \arctan\frac{u}{2} + C = \arctan\frac{x-2}{2} + C$$

$$\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}\,dx$$

$$u = x-2$$

$$= \int \frac{1}{u^2}\,du = -u^{-1}+C = -\frac{1}{x-2}+C$$

$$\int \frac{1}{x^2+2x-3}\,dx = \int \frac{1}{(x-1)(x+3)}\,dx$$

$$x^2+2x-3 = x^2+2\cdot x\cdot 1+1-4$$

$$= (x+1)^2-2^2 = (x+1-2)(x+1+2) = (x-1)(x+3)$$

$$\boxed{a^2-b^2=(a-b)(a+b)}$$

$$\frac{1}{(x-1)(x+3)} = \frac{A}{x-1}+\frac{B}{x+3}$$, A, B to be determined

Multiply by $(x-1)(x+3)$

$1 = A(x+3)+B(x-1)$ for all x

$x=1$: $1 = 4A$, $A=\frac{1}{4}$

$x=-3$: $1 = B\cdot(-4)$, $B=-\frac{1}{4}$

$$\int \frac{1}{x^2+2x-3}\,dx = \int \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+3}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C$$

IN GENERAL:

$$\int \frac{1}{ax^2+bx+c}\,dx$$

ax^2+bx+c

- $a[(x+r)^2+z^2]$ ①
- $a(x+r)^2$ ②
- $a\cdot[(x+r)^2-z^2]$ ③

① arctan

② $\neq -1$ power of $x+r$

③ partial fractions $\ln$

$$\int \frac{x+5}{x^2-4x+8}\,dx = \int \frac{x+5}{(x-2)^2+2^2}\,dx$$

$$x-2=u,\quad x=2+u$$
$$dx=du$$

$$= \int \frac{2+u+5}{u^2+2^2}\,du = \int \frac{u+7}{u^2+2^2}\,du$$

$$= \int \frac{u}{u^2+2^2}\,du + \int \frac{7}{u^2+2^2}\,du$$

$$u^2+2^2=w$$
$$2u\,du=dw$$

$$\int \frac{u}{u^2+2^2}\,du = \int \frac{\frac{1}{2}\,dw}{w} = \frac{1}{2}\ln|w|+C$$
$$= \frac{1}{2}\ln|u^2+4|+C$$

$$= \frac{1}{2}\ln|u^2+4| + 7\arctan\frac{u}{2}$$

$$= \frac{1}{2}\ln|(x-2)^2+4| + 7\arctan\frac{x-2}{2}$$

$$\int \frac{x+5}{x^2+2x+1}\,dx = \int \frac{x+5}{(x+1)^2}\,dx$$

$$x+1=u$$

$$= \int \frac{u+4}{u^2}\,du = \int \frac{1}{u}+\frac{4}{u^2}\,du$$

$$= \ln|u| - \frac{4}{u} + C = \ln|x+1| - \frac{4}{x+1} + C$$

$$\int \frac{x+5}{x^2+2x-3}\,dx = \int \frac{x+5}{(x-1)(x+3)}\,dx$$

$$\frac{x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$x+5 = A(x+3)+B(x-1)$$

$x=1$, $x=-3$, compute A, B