RATIONAL FUNCTIONS

LAST TIME: $\frac{mx+n}{ax^2+bx+c}$, $a \neq 0$

TO INTEGRATE, THREE CASES:

1) $ax^2+bx+c = a(x-r_1)(x-r_2)$

PARTIAL FRACTIONS

$$\frac{mx+n}{a(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$ ln

2) $ax^2+bx+c = a(x-r)^2$

$\frac{mx+n}{a(x-r)^2}$, let $u = x-r$

powers of u

3) $ax^2+bx+c = a\cdot[(x-r)^2+d^2]$

let $u = x-r$

$\int \frac{(\quad)u+(\quad)}{u^2+d^2}\,du$

DEF: $\frac{P(x)}{Q(x)}$ proper if $\deg P < \deg Q$

Use long division of polynomials

$P(x) = Q(x) \cdot M(x) + R(x)$, $\deg R < \deg Q$

$$\frac{P(x)}{Q(x)} = \frac{Q(x)\cdot M(x) + R(x)}{Q(x)} = M(x) + \frac{R(x)}{Q(x)}$$

$M(x)$: easy; $\frac{R(x)}{Q(x)}$: proper

EX: $\int \frac{x^2+1}{x+1}\,dx = \int \frac{(x+1)(x-1)+2}{x+1}\,dx$

$$\begin{array}{r} x-1 \\ x+1 \overline{)\, x^2+0\cdot x+1} \\ \underline{x^2+1\cdot x} \\ -x+1 \\ \underline{-x-1} \\ 2 \end{array}$$

$$= \int (x-1) + \frac{2}{x+1}\,dx$$

$$= \frac{x^2}{2} - x + 2\ln|x+1| + C$$

HOW TO INTEGRATE ANY PROPER RATIONAL FUNCTION $\frac{P(x)}{Q(x)}$?

THM: Any polynomial $Q(x)$ can be factored $Q(x) = A(x-r_1)^{s_1}\cdots(x-r_k)^{s_k} \cdot (x^2+b_1x+c_1)^{p_1}\cdots(x^2+b_\ell x+c_\ell)^{p_\ell}$

EX 1: $Q(x)$ = product of distinct, linear factors

$$\int \frac{x^2+1}{x^3-x}\,dx = \int \frac{x^2+1}{x(x-1)(x+1)}\,dx$$

$$x^3-x = x(x^2-1) = x(x-1)(x+1)$$

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad | \cdot x(x-1)(x+1)$$

$$x^2+1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$x=1$: $2 = B\cdot 1\cdot(1+1)$, $B=1$

$x=0$: $1 = A\cdot(-1)\cdot 1$, $A=-1$

$x=-1$: $2 = C\cdot(-1)\cdot(-1-1)$, $C=1$

$$\int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\,dx$$

$$= -\ln|x| + \ln|x-1| + \ln|x+1| + C$$

$$= \ln\frac{|(x-1)(x+1)|}{|x|} + C$$

<u>EX 2</u>: $Q(x)$ = product of linear factors, some repeated

$$\int \frac{1}{x^3-2x^2+x}\,dx$$

$$x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$1 = A(x-1)^2 + Bx(x-1) + Cx$$

$x=1$: $1 = C$

$x=0$: $1 = A\cdot(-1)^2$, $1 = A$

LOOK AT x^2

$$0 = A + B, \quad B = -1$$

$$\int \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

EX 3: $$\int \frac{5x+2}{(x-1)(x^2+4x+5)}\,dx$$

$$\frac{5x+2}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$$

$$5x+2 = A(x^2+4x+5) + (Bx+C)(x-1)$$

$x=1$: $A = \frac{7}{10}$

x^2

$0 = A + B$, $B = -\frac{7}{10}$

x

$5 = 4A - B + C$, $C = \ldots$

EX 4: $\int \frac{7x - 5}{(x+5)(x-1)^3(x^2+2x+4)^2}\,dx$

$$\frac{A}{x+5} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+2x+4} + \frac{Gx+H}{(x^2+2x+4)^2}$$