APPROXIMATE INTEGRATION

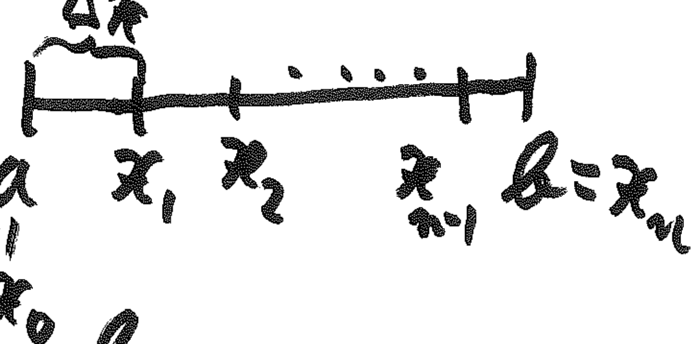
$\int e^{-x^2}dx$, $\int \sin(x^2)dx$

CANNOT BE COMPUTED EXACTLY!

$\int_{-1}^{1} e^{-x^2}dx = ??$

RIEMANN SUMS

$\int_a^b f(x)dx$

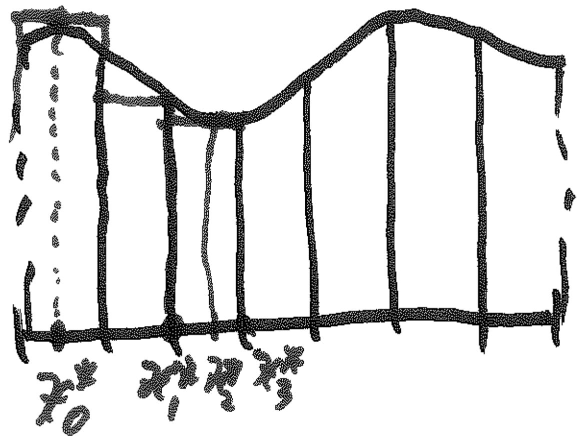


$$\Delta x = x_{k+1} - x_k = \frac{b-a}{n}$$

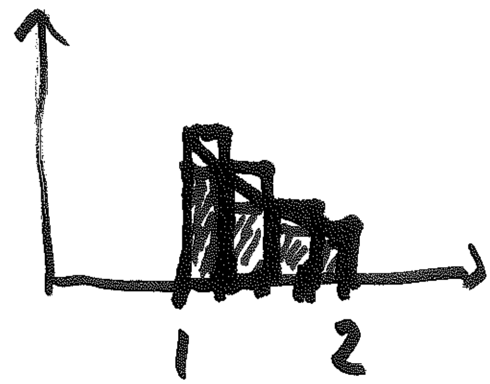
Pick x_k^* so that $x_k \le x_k^* \le x_{k+1}$

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n-1} f(x_k^*)(x_{k+1} - x_k)$$

$$= \Delta x \sum_{k=0}^{n-1} f(x_k^*)$$



<u>EX:</u> $\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx 0.693$



overestimates

underestimates

Left Riemann sum L_n

$n=4$ $\quad L_n = \frac{1}{4}\left(f(1)+f\left(\frac{5}{4}\right)+f\left(\frac{6}{4}\right)+f\left(\frac{7}{4}\right)\right)$

$= \frac{1}{4}\left(\frac{1}{1}+\frac{1}{\frac{5}{4}}+\frac{1}{\frac{6}{4}}+\frac{1}{\frac{7}{4}}\right) = \ldots$

Right Riemann sums R_n

$n=4 \quad R_n = \frac{1}{4}\left(f\left(\frac{5}{4}\right)+f\left(\frac{6}{4}\right)+f\left(\frac{7}{4}\right)+f(2)\right)$

MIDPOINT: $x_k^* = \dfrac{x_k + x_{k+1}}{2}$

$$n=4 \quad M_n = \frac{1}{4}\left(f\left(\frac{1+\frac{5}{4}}{2}\right)+f\left(\frac{\frac{5}{4}+\frac{6}{4}}{2}\right)\right.$$
$$\left. + f\left(\frac{\frac{6}{4}+\frac{7}{4}}{2}\right)+f\left(\frac{\frac{7}{4}+2}{2}\right)\right)$$

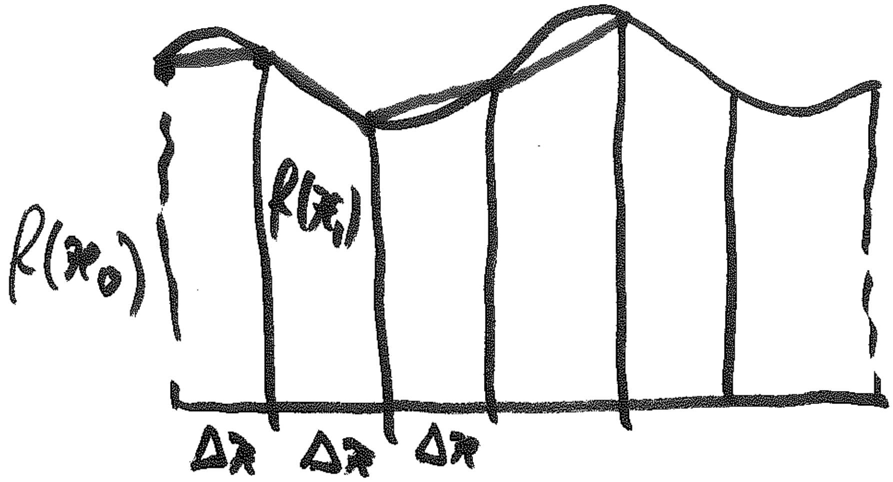
If f is decreasing, then $L_n \geq \int_a^b f\,dx \geq R_n$

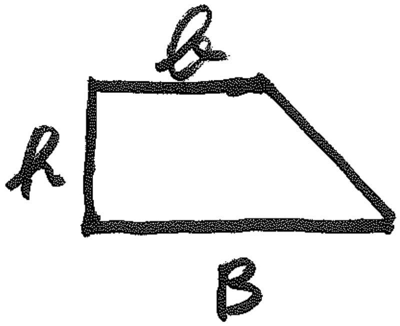
If f is increasing, then $L_n \leq \int_a^b f\,dx \leq R_n$

SO FAR, WE ONLY USED RECTANGLES

i) TRAPEZOID RULE (linear approximation)

ii) SIMPSON'S RULE (quadratic approximation)





$$\text{Area} = h \cdot \frac{B+b}{2}$$

1st trapezoid: $\Delta x \cdot \dfrac{f(x_0)+f(x_1)}{2}$

2nd trapezoid: $\Delta x \cdot \dfrac{f(x_1)+f(x_2)}{2}$

$\vdots$

ADD ALL OF THEM:

$$\int_a^b f(x)\,dx \approx \frac{(\Delta x)}{2}\Big[f(x_0)+2f(x_1)+2f(x_2)+ \\ +\dots+2f(x_{n-1})+f(x_n)\Big]$$

$$\underbrace{}_{= T_n}$$

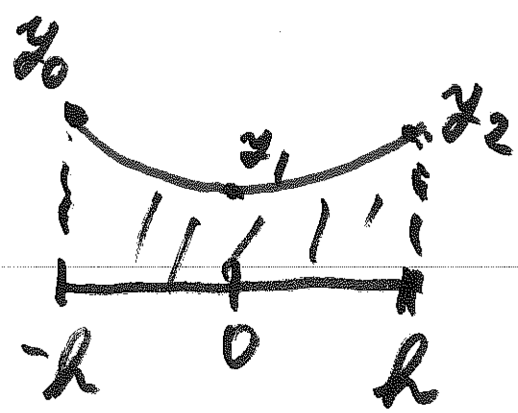
$$\int_1^2 \frac{1}{x}\,dx$$

$$T_4 = \frac{1}{8}\left(\frac{1}{1} + 2\cdot\frac{1}{\frac{5}{4}} + 2\cdot\frac{1}{\frac{6}{4}} + 2\cdot\frac{1}{\frac{7}{4}} + \frac{1}{2}\right)$$

$$= \dots$$

SIMPSON'S RULE

Q: FIND A PARABOLA $f(x) = ax^2 + bx + c$ so that $f(-h) = y_0$, $f(0) = y_1$, $f(h) = y_2$



$$\int_{-h}^{h} ax^2 + bx + c\,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big|_{-h}^{h}$$

$$= \frac{2a}{3}h^3 + 2ch$$

$$f(-h)=y_0 \qquad a(-h)^2+b\cdot(-h)+c=y_0$$

$$f(0)=y_1 \qquad c=y_1$$

$$f(h)=y_2 \qquad ah^2+bh+c=y_2$$

ADD

$$2ah^2+2c=y_0+y_2$$

$$2ah^3+2ch=(y_0+y_2)h \quad \Big|\cdot\frac{1}{3}$$

$$\frac{2a}{3}h^3+\frac{2c}{3}h=\frac{1}{3}h(y_0+y_2) \quad \Big|+\frac{4c}{3}h$$

$$\boxed{\frac{2a}{3}h^3+2ch=\frac{1}{3}h(y_0+4y_1+y_2)}$$