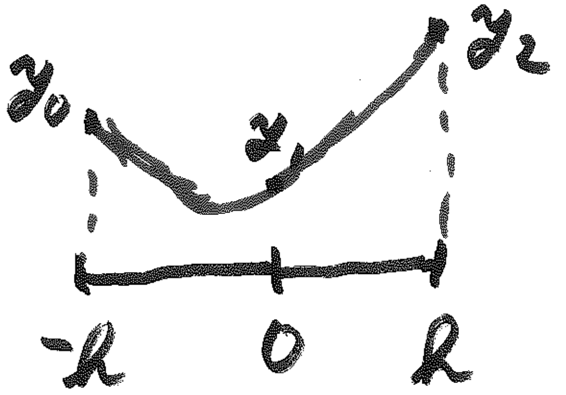
# SIMPSON'S RULE



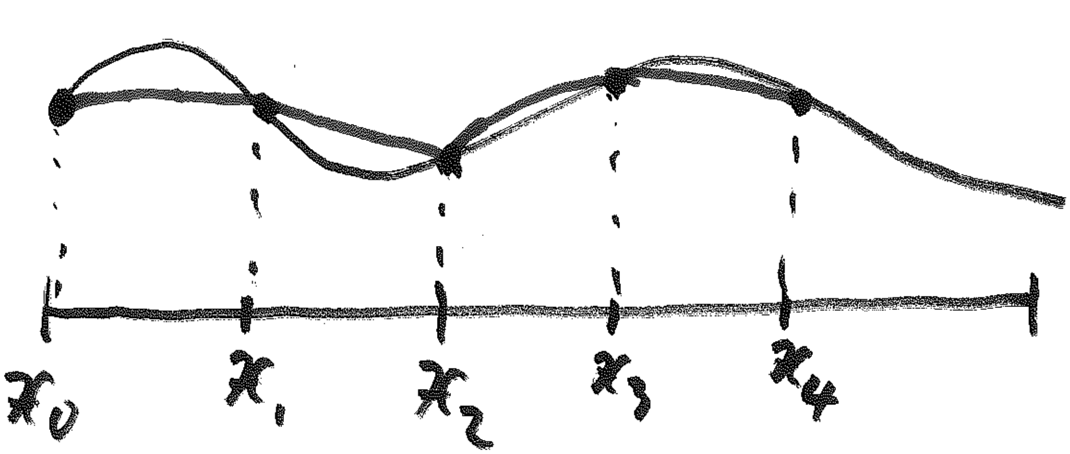

THERE IS EXACTLY ONE QUADRATIC

$f(x) = ax^2 + bx + c$ so that

$f(-h) = y_0 \qquad f(h) = y_2$

$f(0) = y_1$

$$\int_{-h}^{h} f(x)\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$




$h = \Delta x \quad , y_0 = f(x_0) \, , y_1 = f(x_1) \, , y_2 = f(x_2)$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)$$
$$+\frac{\Delta x}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)$$
$$\vdots$$
$$+\frac{\Delta x}{3}\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[&f(x_0)+4f(x_1)+2f(x_2)\\ &+4f(x_3)+2f(x_4)+\dots+4f(x_{n-1})+\\ &+f(x_n)\Big]\end{aligned}\quad n \text{ even}}$$

EX: $\int_1^2 \frac{1}{x}\,dx$, $n=4$, $\Delta x = \frac{1}{4}$, $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{6}{4}$, $x_3 = \frac{7}{4}$, $x_4 = 2$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{12}\left[\frac{1}{1}+4\cdot\frac{1}{\frac{5}{4}}+2\cdot\frac{1}{\frac{6}{4}}+4\cdot\frac{1}{\frac{7}{4}}+\frac{1}{2}\right]$$

# SPEEDOMETER

| $t$ (min) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| speed (mi/h) | 0 | 20 | 25 | 20 | 40 | 35 | 40 |

SIMPSON'S RULE:

$$\text{total distance} \approx \frac{\Delta t}{3}\cdot\big(f(0)+4f(1)+2f(2)$$
$$+4f(3)+2f(4)+4f(5)+f(6)\big)$$

$$\Delta t = 1\,\text{min} = \frac{1}{60}\,h$$

$$\frac{1}{180}\cdot(0+4\cdot 20+2\cdot 25+4\cdot 20+2\cdot 40+4\cdot 35+40)$$

# ESTIMATING THE ERROR

A function $f(x)$ has an upper bound $K$ on $[a,b]$ if $f(x) \leq K$ on $[a,b]$

<u>EX</u>: $f(x) = x^2$ on $[-1, 2]$

$K = 4$ best upper bound

$K = 100$

## ERRORS FOR THE MIDPOINT AND TRAPEZOID RULES

If $I = \int_a^b f(x)\,dx$, then:

$$E_M = |I - M_n| \leq \frac{K(b-a)^3}{24n^2}$$

$$E_T = |I - T_n| \leq \frac{K(b-a)^3}{12n^2}$$

$K$ is any upper bound for $|f''(x)|$

EX: $\int_0^1 e^{-x^2}\,dx$, MIDPOINT METHOD

Q: WHAT IS THE MINIMUM $n$ so that the error is less than $\frac{1}{1000}$?

STEP 1: Compute $f''(x) = (4x-2)e^{-x^2}$

$|f''(x)| \leq 2$ $\rightsquigarrow$ optimal upper bound

$K = 2$

$$E_M \leq \frac{2 \cdot (1-0)^3}{24n^2} \overset{?}{<} \frac{1}{1000}$$

$$\frac{2}{24n^2} < \frac{1}{1000}, \quad 2000 < 24n^2$$

$$83.3 \approx \frac{2000}{24} < n^2, \quad n = 10$$

# ERROR FOR SIMPSON

$$E_S = |I - S_n| \le \frac{\tilde{K}(b-a)^5}{180\, n^4}$$

$\tilde{K}$ is an upper bound for $|f^{(4)}(x)|$

<u>EX</u>: $\int_1^2 \frac{1}{x}\,dx$

$\tilde{K} = 24$