IMPROPER INTEGRALS

If f continuous on $[a, b]$, then

$\int_a^b f(x)\,dx$ is always a finite number.

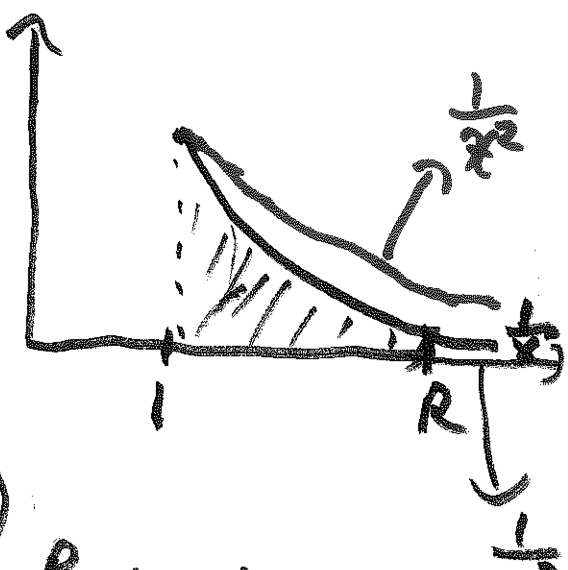
What if $a = -\infty$, or $b = +\infty$, or

$f(c) = \pm\infty$ at some point?

TYPE 1: f continuous and $a = -\infty$ or $b = +\infty$ or both

TYPE 2: a and b are finite, but f becomes $\pm\infty$ at some point

TYPE 1 $\int_1^\infty \frac{1}{x^3}\,dx$



$$\int_1^R \frac{1}{x^3}\,dx \stackrel{FTC}{=} \frac{1}{-2}x^{-2}\Big|_1^R$$

$$= -\frac{1}{2}R^{-2} - \left(-\frac{1}{2}1^{-2}\right)$$

$$= \frac{1}{2} - \frac{1}{2R^2} \qquad \lim_{R\to+\infty} \int_1^R \frac{1}{x^3}\,dx = \frac{1}{2}$$

DEF : i) If $\lim_{R\to+\infty} \int_a^R f(x)\,dx$ exists and is finite, then $\int_a^\infty f(x)\,dx = \lim_{R\to+\infty} \int_a^R f\,dx$

ii) If $\lim_{R\to-\infty} \int_R^b f(x)\,dx$ exists, finite then $\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx$

If i) or ii) holds, then the integral is convergent ; if not, divergent

iii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx$

need to converge

EX : $\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2}$

$\int_1^\infty \frac{1}{x^2}\,dx = 1$

$\int_1^\infty \frac{1}{x}\,dx = +\infty$ diverges

$\lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx \overset{\lim}{=} \ln|x|\Big|_1^R \overset{\lim}{=} \ln R - \cancel{\ln 1}^{\,0} = +\infty$

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \to \text{converges, if } p>1 \\ \to \text{diverges, if } p\le 1 \end{cases}$$

$$\int_1^\infty \overset{\to\ \text{divergent}}{\cos x}\,dx = \lim_{R\to\infty}\int_1^R \cos x\,dx$$

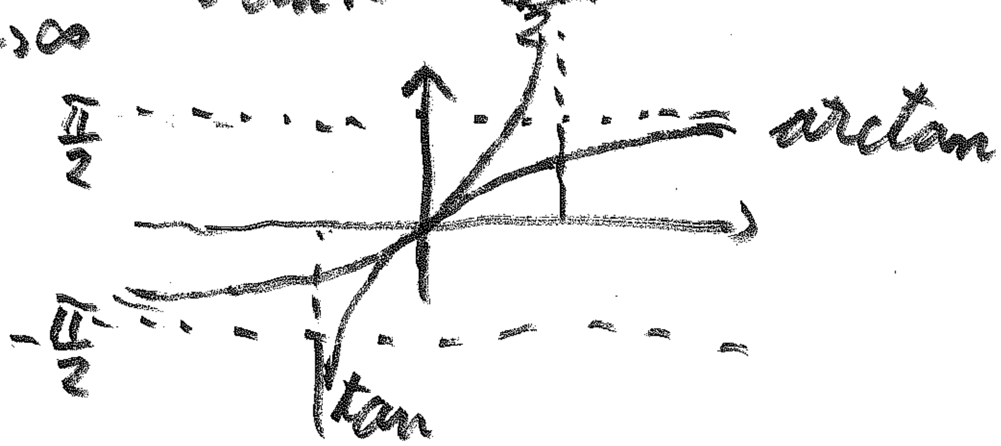
$$= \lim_{R\to\infty} \sin x\Big|_1^R = \lim_{R\to\infty}(\sin R - \sin 1)\ \text{DNE}$$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{1}\frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{1+x^2}\,dx$$

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \arctan x\Big|_1^R$$

$$= \lim_{R\to\infty}(\arctan R - \arctan 1) = \frac{\pi}{2} - \arctan 1$$

$$\lim_{R\to\infty} \arctan R = \frac{\pi}{2}$$



$$\int_{-\infty}^{1} \frac{1}{1+x^2}\,dx = \arctan 1 + \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

TYPE 2: i) If f is continuous on $[a,b)$ discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx$$

ii) If f continuous $(a,b]$, discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx$$

iii) If f continuous on $[a,b]$ except at some point $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \int_R^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{R\to 0^+} 2x^{\frac{1}{2}}\Big|_R^1 = \lim_{R\to 0^+}\left(2 - 2\overset{\to 0}{R^{\frac{1}{2}}}\right) = 2$$

$\int_0^1 \frac{1}{x^2}\,dx$ diverges

IN GENERAL,

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \to \text{converges}, & p<1 \\ \to \text{diverges}, & p\geq 1 \end{cases}$$

CAREFUL!!

$$\int_0^{\pi} \tan x\,dx = \ln|\sec x|\Big|_0^{\pi} = \ln 1 - \ln 1 = 0$$

WRONG!

$\tan\frac{\pi}{2}$ is infinite!

$\int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$ diverges!

WHAT IF WE CAN'T INTEGRATE?

<u>EX</u>: $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$, $\int_1^\infty e^{x^2}\,dx$

<u>COMPARISON THM</u>

f, g continuous, $f(x) \geq g(x) \geq 0$

a) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges

b) If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)$ diverges

$$\underbrace{\frac{\sin^2 x}{x^2}}_{g(x)} \leq \underbrace{\frac{1}{x^2}}_{f(x)}$$

$\int_1^\infty \frac{1}{x^2}\,dx$ converges

$$1 \leq e^{x^2}$$