ON THE INTERACTION OF METRIC TRAPPING AND A BOUNDARY

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Abstract. By considering a two ended warped product manifold, we demonstrate a bifurcation that can occur when metric trapping interacts with a boundary. In this highly symmetric example, as the boundary passes through the trapped set, one goes from a nontrapping scenario where lossless local energy estimates are available for the wave equation to the case of stably trapped rays where all but a logarithmic amount of decay is lost.

1. **Introduction.** We explore the interaction of metric trapping and a boundary in an explicit example and note an extreme bifurcation in the behavior of the local energy for the wave equation as the boundary passes through the trapping. This is closely related to the instability of ultracompact neutron stars as was examined in [**?**]. Here, we instead examine a certain class of exterior domains with Dirichlet boundary conditions on a warped product background geometry and provide a more elementary argument.

For the Minkowski wave equation $\Box = \partial_t^2 - \Delta$, we have a conserved energy $E_0[u](t) = \frac{1}{2}\|\partial u(t,\,\cdot\,)\|_{L^2}^2$ for solutions to the homogeneous wave equation. Here $\partial u = (\partial_t u, \nabla_x u)$ denotes the space-time derivative. One common and robust measure of dispersion is called the (integrated) local energy estimate, which involves examining the energy within a compact set. Specifically if we set

$$\|u\|_{LE[0,T]} = \sup_{j\geq 0} 2^{-j/2}\|u\|_{L^2_tL^2_x([0,T]\times\{\langle x\rangle\approx 2^j\})}, \quad \|u\|_{LE^1[0,T]} = \|(\partial u, \langle x\rangle^{-1}u)\|_{LE[0,T]}$$

and

$$\|F\|_{LE^*[0,T]} = \sum_{j\geq 0} 2^{j/2}\|F\|_{L^2_tL^2_x([0,T]\times\{\langle x\rangle\approx 2^j\})},$$

we have

$$\|\partial u\|_{L^\infty_tL^2_x} + \|u\|_{LE^1} \lesssim \|\partial u(0,\,\cdot\,)\|_{L^2} + \|\Box u\|_{L^1_tL^2_x+LE^*}$$

on $(1+3)-$dimensional Minkowski space. Here LE^1 and LE^* are understood to denote $LE^1[0,\infty)$, $LE^*[0,\infty)$. Such estimates originated in [**?**, **?**]. See, e.g., [**?**] for some of the most general results and a more complete history.

On nonflat geometries, the null geodesics, which packets of the solution tend to flow along, are no longer necessarily straight lines. And in certain geometries, null geodesics may stay in a compact set for all times, and when this happens, trapping is said to occur.

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Trapping is a known obstruction to local energy estimates. In fact, [?], [?] show that the local energy estimate as stated above cannot hold when trapped rays exist.

When the trapping is sufficiently unstable, it is often the case that local energy estimates may be recovered with a small loss, which is often realized as a loss of regularity. This is what happens, e.g., on the Schwarzschild space-times [?]. There it is shown that a logarithmic loss of regularity suffices. See [?, ?] for the seminal results in this direction. When, however, the trapping is elliptic (i.e. stable), it is known that nearly all decay is lost. See, e.g., [?], [?], [?]. In both of these cases, numerous related results have followed. See, e.g., [?, Chapter 6]. The surfaces that we consider are from [?], [?] where they were shown to generate an example of trapping for which an algebraic loss of regularity is both necessary and sufficient.

The notion of being nontrapping is generally known to be stable in the sense that a sufficiently small perturbation of a nontrapping metric remains nontrapping. Here, however, we show that a drastic bifurcation can happen when metric trapping interacts with a boundary. Specifically, on the surfaces used in [?], [?], we shall demonstrate that lossless local energy estimates are available when a boundary exists on one side of the trapping. But as soon as that boundary passes through the trapped set, the interaction with the metric causes stable trapping to form. In this setting, we demonstrate that at most a logarithmic amount of decay is available and no loss of regularity is sufficient in order to recover a local energy estimate.

Specifically we consider the warped products that were examined in [?], [?]. That is, we examine $\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2$ with

$$ds^2 = -dt^2 + dx^2 + a(x)^2\, d\sigma^2_{\mathbb{S}^2}, \quad a(x) = (x^{2m}+1)^{1/2m}, \quad m\in\mathbb{N}.$$

Here $(\mathbb{S}^2, d\sigma^2_{\mathbb{S}^2})$ is the two-dimensional round sphere. This geometry is asymptotically flat on both of its ends, and trapping occurs at $x=0$. When $m\geq 2$, the trapping is degenerate, while for $m=1$ the trapping resembles that of the Schwarzschild metric. We use M_{x_0} to denote the space $(x_0,\infty)\times\mathbb{S}^2$ equipped with the metric $dx^2 + a(x)^2 d\sigma^2_{\mathbb{S}^2}$. We will set $dV = a(x)^2\, dx\, d\sigma_{\mathbb{S}^2}$, while the volume form of the whole space-time $\mathbb{R}_+\times M_{x_0}$, then, is $dV\, dt$. The arguments of these paper should also apply if $\mathbb{S}^2$ is replaced by other compact manifolds, but we will not pursue it here.

On this background (and in these coordinates), the wave equation is given by

$$\Box_{\mathfrak{g}} u = -\partial_t^2 u + \Delta_{\mathfrak{g}} u = -\partial_t^2 u + a(x)^{-2}\partial_x\Big[a(x)^2\partial_x u\Big] + a(x)^{-2}\not{\Delta}_{\mathbb{S}^2} u.$$

We consider the boundary value problem Dirichlet boundary conditions

$$\begin{aligned} \Box_{\mathfrak{g}} u &= F(t,x,\omega), \qquad (t,x,\omega)\in\mathbb{R}_+\times\{x\geq x_0\}\times\mathbb{S}^2,\\ u(t,x_0,\omega) &= 0,\\ u(0,x,\omega) = u_0(x,\omega), &\quad \partial_t u(0,x,\omega) = u_1(x,\omega). \end{aligned} \tag{1.1}$$

Our methods do not directly apply for other boundary conditions. For example, in the proof of Theorem 1.1 for Neumann boundary conditions, there will be an extra boundary term on $x=x_0$ with the wrong sign.

This static space-time and the Dirichlet boundary conditions naturally yield a coercive conserved energy for solutions to the homogeneous equation ($F \equiv 0$)

$$E[u](t) = \frac{1}{2}\int_{x\geq x_0} (\partial_t u)^2 + (\partial_x u)^2 + a(x)^{-2}|\nabla\!\!\!/_0 u|^2\,dV,$$

where $\nabla\!\!\!/_0$ denotes the derivatives tangent to the unit sphere. More generally, we have

$$E[u](t) \leq E[u](0) + \int_0^t \int_{x\geq x_0} |\Box_{\mathfrak{g}} u||\partial_t u|\,dV\,dt. \tag{1.2}$$

We first consider the case of $x_0 > 0$. In this realm, the trapping is not observed and the effect of the boundary is akin to the case of star-shaped obstacles as was examined in [**?**]. We note that, in this case, we can significantly simplify the argument of [**?**] and indeed select a single multiplier that will yield the result rather than needing to consider high and low frequency regimes separately.

Theorem 1.1. *If $x_0 > 0$, then solutions to the wave equation* (1.1) *satisfy the lossless local energy estimate*[1]

$$\|u\|_{LE^1}^2 + \sup_t E[u](t) \lesssim E[u](0) + \|\Box_{\mathfrak{g}} u\|^2_{L^1_t L^2_{M_{x_0}} + LE^*}. \tag{1.3}$$

For each $R > 0$, we shall consider the local energy

$$E_R[u](t) = \frac{1}{2}\int_{x_0\leq x\leq R} (\partial_t u)^2 + (\partial_x u)^2 + a(x)^{-2}|\nabla\!\!\!/_0 u|^2\,dV.$$

We shall use

$$\|(u_0,u_1)\|_{D(B^k)} := \|(u_0,u_1)\|_{H_{x_0}} + \|B^k(u_0,u_1)\|_{H_{x_0}}$$

with

$$H_{x_0} := \dot H^1_0(M_{x_0}) \oplus L^2(M_{x_0}), \quad B := \begin{bmatrix} 0 & iI \\ i\Delta_{\mathfrak{g}} & 0\end{bmatrix}.$$

Our second theorem then says that when $x_0 < 0$ all but a logarithmic amount of decay is lost no matter what loss of regularity k is permitted.

Theorem 1.2. *Let $x_0 < 0$, and fix $R > 0$. Then for any $k \in \mathbb{N}$, if u solves* (1.1) *with $F \equiv 0$,*

$$\limsup_{t\to\infty} \Big(\log^k(t) \sup_{u_0,u_1} \frac{E_R^{1/2}[u](t)}{\|(u_0,u_1)\|_{D(B^k)}}\Big) > 0, \tag{1.4}$$

where the supremum is taken over all $u_0, u_1 \in C^\infty(M_{x_0})$ supported in $\{x_0 \leq x < R\}$ that vanish when $x = x_0$.

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[1] The analog of the LE norm here is

$$\|u\|_{LE[0,T]} = \sup_{j\geq 0} 2^{-j/2}\Big(\int_0^T \int_{\{\langle x\rangle\approx 2^j\}\cap\{x\geq x_0\}} \int_{\mathbb{S}^2} |u(t,x,\omega)|^2 a(x)^2\,d\sigma(\omega)\,dx\,dt\Big)^{1/2},$$

with similar adjustments for LE^*.

We note that the lower bound that we obtain here matches up nicely with the decay obtained in [**?**, Théorème 1.1], namely

$$E_R^{1/2}[u](t) \lesssim \frac{\|(u_0,u_1)\|_{D(B^k)}}{\log^k(t+2)}, \quad t \geq 0.$$

Note, however, that the assumptions in [**?**] are not exactly the same as ours, requiring in particular $a(x) = x$ for $x \gg 1$. While we expect a similar result to hold in our context, we do not prove it here.

We also remark that there is no reason that $R > 0$ is required. The choice was made simply to reduce the number of parameters for the sake of clarity.

Due to the presence of stably trapped broken bicharacteristics, the integrated local energy estimate must also fail. While the above morally states that the solution decays at most logarithmically, it does not strictly rule out the possibility of an integrated local energy estimate. Thus, we also prove the following.

Theorem 1.3. *For any $A > 0$ and any $k \in \mathbb{N}$, there exists a $T > 0$ and data $u_0, u_1 \in C^\infty(M_{x_0})$, which are supported in $\{x_0 \leq x < R\}$ and vanish when $x = x_0$, so that the solution u to* (1.1) *when $F \equiv 0$ satisfies*

$$\|u\|_{LE^1[0,T]} > A\|(u_0,u_1)\|_{D(B^k)}. \tag{1.5}$$

2. **Proof of Theorem 1.1: Nontrapping with a star-shaped boundary.** By standard approximation arguments, we may assume that u_0, u_1, and F have spatial support inside a fixed ball. Finite speed of propation implies that $u(t,x,\omega)$ has compact support in x for any t.

For, say, $f, g \in C^2$, integration by parts and the Dirichlet boundary conditions give

$$\begin{aligned}
-\int_0^T\int_{x\geq x_0} \Box_{\mathfrak{g}} u\Big[f(x)\partial_x u + g(x)u\Big]\,dV\,dt &= \int_{x\geq x_0}\partial_t u\Big[f(x)\partial_x u + g(x)u\Big]\,dV\Big|_0^T \\
&+\int_0^T\int_{x\geq x_0}\Big[f'(x)+g(x)-\frac12 a(x)^{-2}\frac{d}{dx}\Big(a(x)^2 f(x)\Big)\Big](\partial_x u)^2\,dV\,dt \\
&+\int_0^T\int_{x\geq x_0}\Big[f(x)\frac{a'(x)}{a(x)}+g(x)-\frac12 a(x)^{-2}\frac{d}{dx}\Big(a(x)^2 f(x)\Big)\Big]a(x)^{-2}|\nabla\!\!\!/_0 u|^2\,dV\,dt \\
&+\int_0^T\int_{x\geq x_0}\Big[-g(x)+\frac12 a(x)^{-2}\frac{d}{dx}\Big(a(x)^2 f(x)\Big)\Big](\partial_t u)^2\,dV\,dt \\
-\frac12\int_0^T\int_{x\geq x_0}\Big(a(x)^{-2}\frac{d}{dx}[a(x)^2 g'(x)]\Big)u^2\,dV\,dt &+ \frac12\int_0^T\int_{\mathbb{S}^2} f(x_0)(\partial_x u)^2|_{x=x_0}a(x_0)^2\,d\sigma_{\mathbb{S}^2}\,dt.
\end{aligned} \tag{2.1}$$

The identity (2.1) can alternatively be seen by applying the Fundamental Theorem of Calculus to

$$\int_0^T\int_{x\geq x_0}\Big(\partial_t I_1 + \partial_x I_2 + \nabla\!\!\!/_0\cdot I_3\Big)\,d\sigma_{\mathbb{S}^2}\,dx\,dt$$

where

$$I_1 = -\partial_t u\Big(f(x)\partial_x u + g(x)u\Big)a(x)^2,$$
$$I_2 = \frac{1}{2}\Big((\partial_t u)^2 + (\partial_x u)^2 - a(x)^{-2}|\nabla\!\!\!/_0 u|^2\Big)f(x)a(x)^2 + u\partial_x u g(x)a(x)^2 - \frac{1}{2}g'(x)u^2a(x)^2,$$
$$I_3 = \Big(f(x)\partial_x u + g(x)u\Big)\nabla\!\!\!/_0 u.$$

For $0 < \delta \ll 1$, we set

$$f(x) = \frac{x^2}{a(x)^2}, \quad g(x) = \frac{1}{2}a(x)^{-2}\frac{d}{dx}\Big(a(x)^2 f(x)\Big) - \delta\frac{x^{1+2m}}{(1+x^{2m})^{2+\frac{1}{m}}}.$$

Then we record that, for δ sufficiently small, the coefficient of $(\partial_x u)^2$ satisfies

$$f'(x) + g(x) - \frac{1}{2}a(x)^{-2}\frac{d}{dx}\Big(a(x)^2f(x)\Big) = \frac{2x}{(1+x^{2m})^{1+\frac{1}{m}}} - \delta\frac{x^{1+2m}}{(1+x^{2m})^{2+\frac{1}{m}}}$$
$$\gtrsim \frac{x}{(1+x^{2m})^{1+\frac{1}{m}}},$$

the coefficient of $a(x)^{-2}|\nabla\!\!\!/_0 u|^2$ becomes

$$f(x)\frac{a'(x)}{a(x)} + g(x) - \frac{1}{2}a(x)^{-2}\frac{d}{dx}\Big(a(x)^2f(x)\Big) = \frac{x^{1+2m}}{(1+x^{2m})^{1+\frac{1}{m}}} - \delta\frac{x^{1+2m}}{(1+x^{2m})^{2+\frac{1}{m}}}$$
$$\gtrsim \frac{x^{1+2m}}{(1+x^{2m})^{1+\frac{1}{m}}},$$

the coefficient of $(\partial_t u)^2$ is

$$-g(x) + \frac{1}{2}a(x)^{-2}\frac{d}{dx}\Big(a(x)^2f(x)\Big) = \delta\frac{x^{1+2m}}{(1+x^{2m})^{2+\frac{1}{m}}},$$

and the coefficient of u^2 obeys

$$-\frac{1}{2}a(x)^{-2}\frac{d}{dx}[a(x)^2g'(x)] = \frac{2m\,x^{2m-1}}{(1+x^{2m})^{2+\frac{1}{m}}} + \delta\frac{m(2m+1)x^{2m-1}(x^{4m}-4x^{2m}+1)}{(1+x^{2m})^{4+\frac{1}{m}}}$$
$$\gtrsim \frac{x^{2m-1}}{(1+x^{2m})^{2+\frac{1}{m}}}$$

provided that $\delta > 0$ is sufficiently small. Moreover, we note that the boundary term, which is the last term in (2.1), is nonnegative.

Since $0 \le f(x) \le 1$, the Schwarz inequality and (1.2) give

$$\Big|\int_{x\ge x_0} f(x)\partial_t u\partial_x u\,dV\Big| \lesssim E[u](t) \le E[u](0) + \int_0^t\int_{x\ge x_0}|\Box_{\mathfrak{g}}u||\partial u|\,dV\,dt$$

on any time slice. Similarly, since $g(x) \lesssim a(x)^{-1}$, we have

$$\Big|\int_{x\ge x_0} g(x)u\partial_t u\,dV\Big| \lesssim \Big(\int_{x\ge x_0} a(x)^{-2}u^2\,dV\Big)^{1/2}(E[u](t))^{1/2}.$$

So upon establishing a Hardy-type inequality

$$\int_{x\ge x_0} a(x)^{-2}u^2\,dV \lesssim \int_{x\ge x_0}(\partial_x u)^2\,dV, \tag{2.2}$$

we shall also have

$$\Big|\int_{x\geq x_0} g(x)u\partial_t u\,dV\Big| \lesssim E[u](0) + \int_0^t\int_{x\geq x_0} |\Box_{\mathfrak{g}} u||\partial u|\,dV\,dt$$

on every time slice. In order to prove (2.2), we simply note that $x \leq a(x)$ and integrate by parts, while relying on the Dirichlet boundary conditions, to obtain

$$\begin{aligned}\int_{x\geq x_0} a(x)^{-2}u^2\,dV &= \int_{x\geq x_0} u^2\frac{d}{dx}x\,dx\,d\sigma\\ &\lesssim \int_{x\geq x_0} a(x)^{-1}|u\partial_x u|\,dV \lesssim \|a(x)^{-1}u\|_{L^2}\|\partial_x u\|_{L^2}.\end{aligned}$$

Using the bounds from below for each of the coefficients in (2.1) and the above estimation of the time-boundary terms, a local energy estimate of the form

(2.3)
$$\begin{aligned}\int_0^T\int_{x\geq x_0}\Big(x^{-2m-1}(\partial_x u)^2 + x^{-1}a(x)^{-2}|\nabla\!\!\!/_0 u|^2 + x^{-2m-1}(\partial_t u)^2 + x^{-2m-3}u^2\Big)\,dV\,dt\\ \lesssim E[u](0) + \int_0^T\int_{x\geq x_0}|\Box_{\mathfrak{g}} u|\Big(|\partial u| + a(x)^{-1}|u|\Big)\,dV\,dt\end{aligned}$$

follows, though it does not have the sharp weights as $x\to\infty$.

To get the estimate as stated, we shall pair (2.3) with [**?**, Proposition 2.3], which showed
(2.4)
$$\|u\|^2_{LE^1_{x>R}[0,T]} \lesssim E[u](0) + \int_0^T\int_{x\geq x_0}|\Box_{\mathfrak{g}} u|\Big(|\partial u| + a(x)^{-1}|u|\Big)\,dV\,dt + R^{-2}\|u\|^2_{LE_{x\approx R}[0,T]}$$

provided R is sufficiently large. Here, e.g., $LE^1_{x>R}[0,T]$ denotes the $LE^1[0,T]$ norm restricted to the set $\{(t,x,\omega)\,:\,0\leq t\leq T,\,x>R\}$. In order to prove (2.4), we again use (2.1) but this time with

$$f(x) = \Big(1-\beta\Big(\frac{x}{R}\Big)\Big)\frac{x}{x+\rho},\quad g(x) = \frac{1}{2}a(x)^{-2}\frac{x}{x+\rho}\frac{d}{dx}\Big[\Big(1-\beta\Big(\frac{x}{R}\Big)\Big)a(x)^2\Big],\quad \rho\geq R$$

where $\beta(\rho)$ is a monotonically decreasing cutoff that is $\equiv 1$ for $\rho < 1/2$ and vanishes for $\rho > 1$. See [**?**] for more details.

Due to (2.4), it suffices to control $\|u\|^2_{LE^1_{x_0\leq x\leq R}}$ for which the weights at infinity are irrelevant and (2.3) suffices. Applying the Schwarz inequality to the forcing term and bootstrapping completes the proof.

3. **Proof of Theorem 1.2 and Theorem 1.3: Stable trapping and the nonexistence of integrable local energy decay.**

Here we shall need the following sequence of exponentially good quasimodes.

Proposition 3.1. *There is a constant $c>0$, a sequence of positive numbers $\tau_j\to+\infty$, and functions $v_j\in C^\infty(M_{x_0})$ such that $\operatorname{supp} v_j\subseteq\{x_0\leq x<0\}$, $\|v_j\|_{L^2(M_{x_0})}=1$, $v_j|_{x=x_0}=0$, and for each $k=0,1,2,\ldots$ there exists $C_k>0$ so that*

$$\|(-\Delta_{\mathfrak{g}}-\tau_j^2)v_j\|_{H^k(M_{x_0})}\leq C_k e^{-c\tau_j}. \tag{3.1}$$

Before we proceed to proving the Proposition, we shall first show how these quasimodes can be used to complete the proof of Theorem 1.2. These arguments are akin to those of [?], [?], and others.

Let

$$u_{0,j}(x,\omega) := v_j(x,\omega), \quad u_{1,j}(x,\omega) := -i\tau_j v_j(x,\omega),$$
$$U_j(t) = (U_{0,j}(t), U_{1,j}(t)) := e^{-it\tau_j}(v_j, -i\tau_j v_j) \in H_{x_0}.$$

It follows immediately from the definition of $\|\cdot\|_{D(B^k)}$ and Proposition 3.1 that for each $k \in \mathbb{N}$, there is $C_k > 0$ so that for all $j \in \mathbb{N}$

$$\|(u_{0,j}, u_{1,j})\|_{D(B^k)} = \|(v_j, -i\tau_j v_j)\|_{D(B^k)} \le C_k \tau_j^k \|(v_j, -i\tau_j v_j)\|_{H_{x_0}}. \tag{3.2}$$

One can check that U_j solves the inhomogeneous wave equation

$$\begin{cases} \partial_t U_j + iBU_j = F_j := (0, (-\Delta_{\mathfrak{g}} - \tau_j^2)v_j), & \text{on } \mathbb{R}_+ \times M_{x_0}, \\ U_j(0) = (v_j, -i\tau_j v_j). \end{cases}$$

Next let

$$\tilde{U}_j(t) = (\tilde{U}_{0,j}(t), \tilde{U}_{1,j}(t)) := e^{-itB}(v_j, -i\tau_j v_j)$$

be the solution to the homogeneous equation where $F_j = 0$. Note that if u_j solves (1.1) with $u_0 = u_{0,j}, u_1 = u_{1,j}$, and $F \equiv 0$, then $\tilde{U}_{0,j} = u_j$, $\tilde{U}_{1,j} = \partial_t u_j$. By Duhamel's principle, the relationship between U_j and $\tilde{U}_j$ is

$$U_j(t) = \tilde{U}_j(t) + \int_0^t e^{-i(t-s)B} F_j \, ds.$$

Using (3.1), we estimate

$$\begin{aligned} \Big(\int_{x_0 \le x \le R} |\nabla_{\mathfrak{g}}(U_{0,j} - \tilde{U}_{0,j})(t)|^2 + |(U_{1,j} - \tilde{U}_{1,j})(t)|^2 \, dV \Big)^{1/2} &\le \Big\| \int_0^t e^{-i(t-s)B} F_j \, ds \Big\|_{H_{x_0}} \\ &\le t \|F_j\|_{H_{x_0}} \le C_0 t e^{-c\tau_j}. \end{aligned}$$

Let $J \in \mathbb{N}$ be sufficiently large that $\tau_j \ge 1$ for $j \ge J$. Then for any $t \in [0, t_j]$ where

$$t_j := \frac{1}{2C_0} e^{c\tau_j} \le \frac{1}{2C_0} e^{c\tau_j} \|(v_j, -i\tau_j v_j)\|_{H_{x_0}},$$

it holds that

$$\begin{aligned} E_R^{1/2}[u_j](t) &\ge \|U_j(t)\|_{H_{x_0}} - \Big(\int_{x_0 \le x \le R} |\nabla_{\mathfrak{g}}(U_{0,j} - \tilde{U}_{0,j})(t)|^2 + |(U_{1,j} - \tilde{U}_{1,j})(t)|^2 \, dV \Big)^{1/2} \\ &\ge \|(v_j, -i\tau_j v_j)\|_{H_{x_0}} - C_0 t e^{-c\tau_j} \\ &\ge \frac{1}{2} \|(v_j, -i\tau_j v_j)\|_{H_{x_0}} \\ &\ge \frac{1}{2C_k \tau_j^k} \|(v_j, -i\tau_j v_j)\|_{D(B^k)}. \end{aligned}$$

In the last step, we have used (3.2). We have thus shown

$$\frac{E_R^{1/2}[u_j](t)}{\|(u_{0,j}, u_{1,j})\|_{D(B^k)}} \ge \frac{1}{2C_k \tau_j^k} = \frac{c^k}{2C_k} \log^{-k}(2C_0 t_j), \quad t \in [0, t_j],$$

from which Theorem 1.2 follows immediately.

By integrating the above inequality, we also obtain

$$\begin{aligned}\|u_j\|_{LE^1[0,t_j]} &\ge C_{x_0}\|E_0^{1/2}[u_j](t)\|_{L^2[0,t_j]}\\ &\ge C_{x_0,k}\frac{t_j^{1/2}}{\log^k(t_j)}\|(u_{0,j},u_{1,j})\|_{D(B^k)}.\end{aligned}$$

Since $\tau_j \to \infty$ as $j \to \infty$, given any $A > 0$, we can select j sufficiently large so $C_{x_0,k}\frac{t_j^{1/2}}{\log^k(t_j)} > A$, which proves Theorem 1.3.

3.1. *Proof of Proposition 3.1:* By expanding into spherical harmonics, we can write

$$a(x)(-\Delta_{\mathfrak{g}})a(x)^{-1} = \bigoplus_{j=0}^{\infty}\Big(-\frac{d^2}{dx^2} + \sigma_j^2 a(x)^{-2} + a''(x)a(x)^{-1}\Big)$$

where $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \dots$ are the square roots of the eigenvalues of the Laplacian on $\mathbb{S}^2$, repeated according to multiplicity. We set

$$V_j(x) := \sigma_j^2 a(x)^{-2} + a''(x)a(x)^{-1}, \quad V_0 := a''(x)a(x)^{-1}.$$

We will show that there is a sequence $\tau_j \to +\infty$ so that we have $u_j \in C_c^\infty([x_0, 0))$, $\|u_j\|_{L^2(\mathbb{R})} = 1$, $u(x_0) = 0$, and for each $k = 0, 1, 2, \dots$

$$\Big(-\frac{d^2}{dx^2} + V_j(x) - \tau_j^2\Big)u_j(x) = \mathcal{O}_{H^k((x_0,0))}(e^{-c\tau_j}). \tag{3.3}$$

This will imply (3.1) with $v_j(x,\omega) = a(x)^{-1}u_j(x)Y_j(\omega)$ where Y_j is a spherical harmonic with eigenvalue σ_j^2.

The first lemma fixes the sequence τ_j.

Lemma 3.2. *For j large enough, V_j is strictly increasing on $[x_0, x_0/2]$, we have $V_j(x_0/2) < V_j(0)$, and the operator $P := -\frac{d^2}{dx^2} + V_j(x)$ on $(x_0, 0)$ with Dirichlet boundary conditions has an eigenvalue $\tau_j^2 \in [V_j(x_0), V_j(x_0/2)]$.*

Proof. To prove that, for j large enough, we have V_j strictly increasing on $[x_0, x_0/2]$ and $V_j(x_0/2) < V_j(0)$, it suffices to use the fact that $\sigma_j^2 a(x)^{-2}$ is strictly increasing on $[x_0, 0]$ when $j > 0$ and that $\sigma_j \to \infty$ as $j \to \infty$.

Let $I = (x_0, 0)$, let $\mathcal{D} = H_0^1(I) \cap H^2(I)$ be the domain of P, and let τ_j^2 be the first eigenvalue of P. We prove the upper bound on τ_j^2 by comparing the bottom of the spectrum of P with the bottom of the spectrum of an explicit infinite square well, whose first eigenvalue can be directly calculated. Let

$$P_+ = -\frac{d^2}{dx^2} + V_j(3x_0/4), \quad \text{have domain } \mathcal{D}_+ = H_0^1(I_+) \cap H^2(I_+)$$

where $I_+ = (x_0, 3x_0/4)$. We then have

$$\begin{aligned}
\tau_j^2 = \inf_{u\in\mathcal{D}} \frac{\langle Pu,u\rangle_{L^2(I)}}{\|u\|^2_{L^2(I)}} &\leq \inf_{u\in C_0^\infty(I_+)} \frac{\langle Pu,u\rangle_{L^2(I)}}{\|u\|^2_{L^2(I)}} \\
&\leq \inf_{u\in C_0^\infty(I_+)} \frac{\langle P_+u,u\rangle_{L^2(I_+)}}{\|u\|^2_{L^2(I_+)}} \\
&= V_j(3x_0/4) + \inf_{u\in C_0^\infty(I_+)} \frac{\|u'\|^2_{L^2(I_+)}}{\|u\|^2_{L^2(I_+)}} = V_j(3x_0/4) + 16\pi^2 x_0^{-2},
\end{aligned}$$

which is bounded by $V_j(x_0/2)$ for j large enough. The last equality follows by computing the smallest constant ρ so $u''+\rho u = 0$ has a nontrivial solution with $u(x_0) = u(3x_0/4) = 0$.

To prove the remaining lower bound, we observe that

$$\inf_{u\in\mathcal{D}} \frac{\langle Pu,u\rangle_{L^2(I)}}{\|u\|^2_{L^2(I)}} \geq V_j(x_0).$$

□

Fix $\chi \in C^\infty([x_0,0];[0,1])$ with $\chi(x) \equiv 1$ on a neighborhood of $[x_0, x_0/2]$ and $\chi(x) \equiv 0$ on a neighborhood of 0. Let $\psi_j \in C^\infty(I)$, $\psi_j(x_0) = \psi_j(0) = 0$ be an eigenfunction associated to the eigenvalue τ_j^2, as supplied by Lemma 3.2. Thus,

$$\Big(-\frac{d^2}{dx^2} + V_j(x) - \tau_j^2\Big)\psi_j = 0, \quad \tau_j^2 \in [V_j(x_0), V_j(x_0/2)]. \tag{3.4}$$

We define the quasimodes to be

$$u_j(x) = \chi(x)\psi_j(x)/\|\chi\psi_j\|_{L^2}.$$

In order to prove (3.3), we will prove the following Agmon estimate, which is a variant of the standard semiclassical Agmon estimate as in [**?**, Section 7.1].

Lemma 3.3. *There exists a constant c so that, for j large enough,*

$$\|\mathbf{1}_{\operatorname{supp}(1-\chi)}\psi_j\|_{L^2([x_0,0])} \leq e^{-c\sigma_j}\|\psi_j\|_{L^2([x_0,0])}. \tag{3.5}$$

Proof. Let $\varphi \in C^\infty([x_0,0])$ such that $\varphi \equiv 0$ on a neighborhood of $[x_0, x_0/2]$ and $\varphi \equiv 1$ on a neighborhood of $\operatorname{supp}(1-\chi)$. We then fix $\chi_0 \in C^\infty([x_0,0])$ with $\chi_0 \equiv 0$ on a neighborhood of $[x_0, x_0/2]$ and $\chi_0 \equiv 1$ on a neighborhood of $\operatorname{supp}\varphi$. Define

$$w := \chi_0 e^{\delta\sigma_j\varphi}\psi_j, \quad \delta > 0 \text{ to be chosen later} \tag{3.6}$$

and

$$\begin{aligned}
P_\varphi :&= e^{\delta\sigma_j\varphi}\Big(-\frac{d^2}{dx^2} + V_j(x) - \tau_j^2\Big)e^{-\delta\sigma_j\varphi} \\
&= e^{\delta\sigma_j\varphi}\Big(-\frac{d^2}{dx^2} + \sigma_j^2 a(x)^{-2} + V_0(x) - \tau_j^2\Big)e^{-\delta\sigma_j\varphi} \\
&= -\frac{d^2}{dx^2} + 2\delta\sigma_j\varphi'\frac{d}{dx} + \sigma_j^2 a(x)^{-2} - \delta^2\sigma_j^2(\varphi')^2 + \delta\sigma_j\varphi'' + V_0(x) - \tau_j^2.
\end{aligned} \tag{3.7}$$

Using $\operatorname{Re} 2\delta\sigma_j\langle\varphi' w', w\rangle = -\delta\sigma_j\langle\varphi'' w, w\rangle$, we compute

$$\begin{aligned}\operatorname{Re}\langle P_\varphi w, w\rangle_{L^2} &= \|w'\|_{L^2}^2 + \operatorname{Re} 2\delta\sigma_j\langle\varphi' w', w\rangle_{L^2} \\ &\quad + \langle(\sigma_j^2(a^{-2} - \delta^2(\varphi')^2) + \delta\sigma_j\varphi'' + V_0 - \tau_j^2)w, w\rangle_{L^2} \\ &= \|w'\|_{L^2}^2 + \langle(\sigma_j^2(a^{-2} - \delta^2(\varphi')^2) + V_0 - \tau_j^2)w, w\rangle_{L^2}.\end{aligned} \tag{3.8}$$

Here and in the sequel we have abbreviated $L^2([x_0, 0])$ by L^2. Since $\tau_j^2 \leq |V_j(x_0/2)| \leq \sigma_j^2 a(x_0/2)^{-2} + |V_0(x_0/2)|$, we now estimate, on $\operatorname{supp}\chi_0$,

$$\begin{aligned}\sigma_j^2(a^{-2} - \delta^2(\varphi')^2) + V_0 - \tau_j^2 \geq \sigma_j^2\Big(\Big(\max_{\operatorname{supp}\chi_0} a^2\Big)^{-1} - a(x_0/2)^{-2} - \delta^2 \max_{\operatorname{supp}\chi_0}(\varphi')^2\Big) \\ - \max_{\operatorname{supp}\chi_0}|V_0| - |V_0(x_0/2)|.\end{aligned} \tag{3.9}$$

Because $a'(x) < 0$ for $x < 0$ and $\operatorname{supp}\chi_0 \subseteq (x_0/2, 0]$, we can fix $\delta > 0$ small enough so that

$$\Big(\max_{\operatorname{supp}\chi_0} a^2\Big)^{-1} - a(x_0/2)^{-2} - \delta^2 \max_{\operatorname{supp}\chi_0}(\varphi')^2 =: \alpha > 0.$$

Then if $\sigma_j > 1$ is sufficiently large, we can ensure that

$$\sigma_j^2(a^{-2} - \delta^2(\varphi')^2) + V_0 - \tau_j^2 \geq \frac{\alpha\sigma_j^2}{2}, \qquad \text{on } \operatorname{supp}\chi_0.$$

Using this and the fact that $\sigma_j^2 > 1$ in (3.8) gives

$$\frac{\alpha}{2}\|w\|_{L^2}^2 \leq \frac{\alpha\sigma_j^2}{2}\|w\|_{L^2}^2 \leq \|P_\varphi w\|_{L^2}\|w\|_{L^2} \leq \frac{1}{\alpha}\|P_\varphi w\|_{L^2}^2 + \frac{\alpha}{4}\|w\|_{L^2}^2.$$

Therefore

$$\|w\|_{L^2}^2 \leq \frac{4}{\alpha^2}\|P_\varphi w\|_{L^2}^2. \tag{3.10}$$

We now use elliptic estimates to show

$$\|P_\varphi w\|_{L^2} \lesssim \sigma_j^2\|\psi_j\|_{L^2}. \tag{3.11}$$

Integration by parts and (3.4) give

$$\begin{aligned}\|\psi_j'\|_{L^2}^2 &\leq \frac{1}{2}\|\psi_j\|_{L^2}^2 + \frac{1}{2}\|\psi_j''\|_{L^2}^2 \\ &\leq \Big(\frac{1}{2} + (\max_{x\in[x_0,0]} V_j(x))^2\Big)\|\psi_j\|_{L^2}^2 \\ &\lesssim \sigma_j^4\|\psi_j\|_{L^2}^2.\end{aligned} \tag{3.12}$$

By (3.4), (3.6), and (3.7),

$$\begin{aligned}P_\varphi w &= \chi_0(P_\varphi e^{\delta\sigma_j\varphi}\psi_j) + [P_\varphi, \chi_0]e^{\delta\sigma_j\varphi}\psi_j \\ &= \chi_0 e^{\delta\sigma_j\varphi}\Big(-\frac{d^2}{dx^2} + V_j - \tau_j^2\Big)\psi_j + [P_\varphi, \chi_0]e^{\delta\sigma_j\varphi}\psi_j \\ &= \Big[-\frac{d^2}{dx^2}, \chi_0\Big]e^{\delta\sigma_j\varphi}\psi_j + \Big[2\delta\sigma_j\varphi'\frac{d}{dx}, \chi_0\Big]e^{\delta\sigma_j\varphi}\psi_j.\end{aligned} \tag{3.13}$$

Using the support properties of φ and χ_0, we get

$$\Big[-\frac{d^2}{dx^2}, \chi_0\Big]e^{\delta\sigma_j\varphi} = \Big[-\frac{d^2}{dx^2}, \chi_0\Big], \quad \Big[2\delta\sigma_j\varphi'\frac{d}{dx}, \chi_0\Big]e^{\delta\sigma_j\varphi} = 0.$$

Since $[-\frac{d^2}{dx^2}, \chi_0] = -\chi_0'' - 2\chi_0' \frac{d}{dx}$, using the triangle inequality along with (3.12) and (3.13) establishes (3.11).

Since $\varphi \equiv \chi_0 \equiv 1$ on $\operatorname{supp}(1-\chi)$, (3.10) and (3.11) show

$$e^{2\delta\sigma_j} \int_{\operatorname{supp}(1-\chi)} |\psi_j|^2\, dx \le \|w\|_{L^2}^2 \lesssim \sigma_j^4 \|\psi_j\|_{L^2}^2,$$

which implies the desired result (3.5). □

We now complete the proof by establishing (3.3):

Proposition 3.4. *There exists a constant c so that for any $k = 0, 1, \dots,$ we have*

$$\Big\| \Big(-\frac{d^2}{dx^2} + V_j - \tau_j^2 \Big) u_j \Big\|_{H^k((x_0,0))} \le C_k e^{-c\sigma_j}.$$

Proof. As (3.5) gives

$$\|(1-\chi)\psi_j\|_{L^2} \le e^{-c\sigma_j} \|\psi_j\|_{L^2} \implies \|\chi\psi_j\|_{L^2} \ge (1 - e^{-c\sigma_j})\|\psi_j\|_{L^2},$$

it suffices to bound the norms of $(-\frac{d^2}{dx^2} + V_j - \tau_j^2)(\chi\psi_j)$ in terms of $\|\psi_j\|_{L^2}$.

Integrating by parts gives

$$\begin{aligned} 2\|\chi'\psi_j'\|_{L^2}^2 &\le -2\int_{x_0}^0 \Big(2\chi'\chi''\psi_j' + (\chi')^2\psi_j'' \Big) \overline{\psi_j}\, dx \\ &\le \|\chi'\psi_j'\|_{L^2}^2 + C \int_{\operatorname{supp}\chi'} |\psi_j|^2\, dx + C \int_{\operatorname{supp}\chi'} \Big| V_j(x) - \tau_j^2 \Big| |\psi_j|^2\, dx. \end{aligned}$$

Noting that $\operatorname{supp}\chi' \subseteq \operatorname{supp}(1-\chi)$, this yields

$$\|\chi'\psi_j'\|_{L^2}^2 \lesssim \sigma_j^2 \int_{\operatorname{supp}(1-\chi)} |\psi_j|^2\, dx, \tag{3.14}$$

provided j is sufficiently large. Using (3.14) along with (3.4) and (3.5), we get

$$\begin{aligned} \Big\| \Big(-\frac{d^2}{dx^2} + V_j - \tau_j^2 \Big) \chi\psi_j \Big\|_{L^2} &= \Big\| \Big[-\frac{d^2}{dx^2}, \chi \Big] \psi_j \Big\|_{L^2} \\ &\le \|\chi''\psi_j\|_{L^2} + 2\|\chi'\psi_j'\|_{L^2} \\ &\le C\sigma_j \|\mathbf{1}_{\operatorname{supp}(1-\chi)}\psi_j\|_{L^2} \\ &\le e^{-c\sigma_j}\|\psi_j\|_{L^2}, \end{aligned}$$

as desired.

The bounds on the higher Sobolev norms follow by an induction argument and using (3.4) and integration by parts repeatedly as above. □

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