THE STRAUSS CONJECTURE ON KERR BLACK HOLE BACKGROUNDS

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Abstract. We examine solutions to semilinear wave equations on black hole backgrounds and give a proof of an analog of the Strauss conjecture on the Schwarzschild and Kerr, with small angular momentum, black hole backgrounds. The key estimates are a class of weighted Strichartz estimates, which are used near infinity where the metrics can be viewed as small perturbations of the Minkowski metric, and a localized energy estimate on the black hole background, which handles the behavior in the remaining compact set.

1. Introduction

In this article, we study an analog of the Strauss conjecture on the Schwarzschild and Kerr, with small angular momentum, black hole backgrounds. In particular, we establish the global existence of solutions to a class of semilinear wave equations with power-type nonlinearities with power greater than a certain critical power. This critical power, $1+\sqrt{2}$, is the same as that on $(1+3)$-dimensional Minkowski space.

More specifically, we will consider the evolution of the nonlinear waves on Kerr black hole backgrounds,

$$\Box_K u = F_p(u), \qquad u|_{\tilde v=0} = f, \qquad \tilde T u|_{\tilde v=0} = g\ . \tag{1.1}$$

Here $\Box_K$ denotes the d'Alembertian in the Kerr metric, and $\tilde T$ is a smooth, everywhere timelike vector field that equals ∂_t away from the black hole. Similarly, the coordinate $\tilde v$ is chosen so that the slice $\tilde v = 0$ is space-like and so that $\tilde v = t$ away from the black hole. A more detailed description of the Kerr geometry is provided in the next section. We shall assume that the nonlinear term behaves like $|u|^p$ when u is small:

$$\sum_{0\le j\le 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p \quad \text{for } |u| \ll 1\ . \tag{1.2}$$

Typical examples include $F_p(u) = \pm|u|^p$ and $\pm|u|^{p-1}u$.

The Strauss conjecture concerned the Minkowski case and determining the values p for which global existence could be guaranteed if the initial data are sufficiently small. The first work [29] showed that small data global existence was available in (1+3)-dimensional

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Minkowski space-time for powers $p > 1+\sqrt{2}$ but blow-up could occur for arbitrarily small data when $p < 1+\sqrt{2}$. Shortly afterward, [53] included the conjecture that the critical power p_c on Minkowski space-time $\mathbb{R}^{n+1}$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0 \ .$$

The existence portion of the conjecture was verified in [23] ($n = 2$), [62] ($n = 4$), [34] ($n \le 8$), and [22], [54] (generic n). The necessity of $p > p_c$ for small data global existence is from [29], [24], [46], [45], [60], and [63].

Some recent works have sought to extend these results to scenarios that include nontrivial background geometry. These include [20] ($n = 4$), [27] ($n = 3, 4$), and [48] ($n = 2$), which examine global existence for similar equations exterior to nontrapping obstacles. See also [61] ($n = 3, 4$) for related results with certain trapping obstacles. In the case of nontrapping aymptotically Euclidean manifolds the same results were obtained in [50] (radial metrics, $n = 3$) and [58] (general metrics $n = 3, 4$).

We seek to show the same on Kerr black hole backgrounds with small angular momentum. In particular, we have

Theorem 1.1. *For Kerr space-times with sufficiently small angular momentum and for initial data which are smooth, compactly supported, and sufficiently small, there exists a global solution u to (1.1) provided that $p > 1+\sqrt{2}$.*

A more precisely stated theorem will be provided in Section 4 after more notation is introduced. In fact, our theorem holds more generally than we state. The proof does not rely on the precise geometry of the Kerr space-time and rather only depends on having a metric which is asymptotically Euclidean and for which there is a sufficiently nice localized energy estimate. The latter will be described further in the next section.

On the Schwarzschild space-times, such nonlinear wave equations have been previously studied for large powers. In particular, see [4] and [43] for related Klein-Gordon equations, [14] ($p > 4$ with radial data), [8] ($p > 3$), and [35] ($p = 5$), though well-known arguments (see, e.g., [49]) allow one to use Strichartz estimates, such as those proved in [35], to prove small data global existence for other $p > 3$. While no such explicit results have been previously given on Kerr backgrounds, the key estimates are known in some cases. For example, [57] provides Strichartz estimates in the case that the angular momentum is small. In the opposite direction, [12] provides blow-up for $p < 1+\sqrt{2}$. The current result fills in the gap $1+\sqrt{2} < p < 3$.

The strategy of proof is to use the weighted Strichartz estimates of [21] and [27] (see also [26] for a radial version, which appeared previously) near infinity where the Kerr metric can be viewed as a small perturbation of the Minkowski metric. See [49] and [59] for more details on the history. In the compact set that remains, localized energy estimates suffice. We rely on the localized energy estimates of [55], though other variants of these are available as will be described in the following section.

In the case of the Kerr metric our proof requires that the initial data have compact support. However, as we shall see in Section 5, we are able to drop this technical assumption in the special case of the Schwarzschild metric. It will be interesting to see whether such results hold for the Kerr metric as well.

For convenience and to ease the exposition, we have taken data on a $\tilde{v} = 0$ slice. Passing from data on $t = 0$ to data on $\tilde{v} = 0$ is a problem of local well-posedness, which can be solved by contraction in energy spaces. We omit these details, though a related argument, which requires Strichartz estimates, can be found in [35].

1.1. **Notation.** The relevant sets of vector fields we shall use are as follows

$$\{\partial_t, \nabla_x\} = \{\partial\}, \quad \Omega = x \wedge \nabla_x \ ,$$
$$Y = \{\nabla_x, \Omega\}, \quad Z = \{\partial, \Omega\} = \{\partial_t\} \cup Y \ .$$

Let $\langle x \rangle = \sqrt{1+|x|^2}$ and L^q_ω be the standard Lebesgue space on the sphere $\mathbb{S}^2$. We will use the following mixed-norm $L^{q_1}_{\tilde{v}} L^{q_2}_r L^{q_3}_\omega$,

$$\|f\|_{L^{q_1}_{\tilde{v}} L^{q_2}_r L^{q_3}_\omega(\mathcal{M})} = \left\| \left(\int_{r_e}^\infty \|f(\tilde{v}, r\omega)\|^{q_2}_{L^{q_3}_\omega} r^2 dr \right)^{1/q_2} \right\|_{L^{q_1}(\{\tilde{v}\ge 0\})} ,$$

with trivial modification for the case $q_2 = \infty$. Occasionally, we will omit the subscripts. We will also use $A \lesssim B$ to denote the inequality $A \le CB$ with some positive constant C, which may change from line to line. We also use the following convention (for invertible functions f and function spaces H)

$$g \in fH \Leftrightarrow f^{-1} g \in H \ .$$

Let us also recall some notations from [5, Section 5.6]. Let A be a Banach space, $s \in \mathbb{R}$ and $q > 0$, we use $l^s_q(A)$ to denote the space of all sequences $(a_j)_{j=0}^\infty$, $a_j \in A$ such that

$$\|(a_j)\|_{l^s_q(A)} = \|2^{js}\|a_j\|_A\|_{l^q_{j\ge 0}} < \infty \ .$$

For a partition of unity subordinate to the dyadic (spatial) annuli, $1 = \sum_{j\ge 0} \phi_j^2(x)$, we shall abuse notation and write

$$\|u\|_{l^s_q(A)} = \|(\phi_j(x) u(t,x))\|_{l^s_q(A)}.$$

2. The Kerr metric and localized energy estimates

Let us first recall the Kerr metric. In Boyer-Lindquist coordinates, it is given by

$$ds^2 = g_{tt}\, dt^2 + g_{t\phi}\, dt\, d\phi + g_{rr}\, dr^2 + g_{\phi\phi}\, d\phi^2 + g_{\theta\theta}\, d\theta^2$$

where $t \in \mathbb{R}$, $r > 0$, (ϕ, θ) are the spherical coordinates on $\mathbb{S}^2$ and

$$g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}, \qquad g_{t\phi} = -2a\frac{2Mr\sin^2\theta}{\rho^2}, \qquad g_{rr} = \frac{\rho^2}{\Delta},$$
$$g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \qquad g_{\theta\theta} = \rho^2$$

with

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

Here M represents the mass of the black hole and aM its angular momentum. The Schwarzschild space-time is the static solution corresponding to $a = 0$. And the Minkowski space-time is the trivial solution to Einstein's equations for which we have $a = 0$ and $M = 0$.

For convenience, we record that

$$dVol = \rho^2 \sin\theta\, dr\, d\theta\, d\phi\, dt = \rho^2\, dr\, d\omega\, dt$$

where ω is the standard measure on $\mathbb{S}^2$. Moreover, the inverse of the metric is given by:

$$g^{tt} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}, \qquad g^{t\phi} = -a\frac{2Mr}{\rho^2\Delta}, \qquad g^{rr} = \frac{\Delta}{\rho^2},$$
$$g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}, \qquad g^{\theta\theta} = \frac{1}{\rho^2}.$$

One can view M as a scaling parameter, and a scales in the same way as M. Thus M/a is a dimensionless parameter. We shall subsequently assume that a is small, $a/M \ll 1$, so that the Kerr metric is a small perturbation of the Schwarzschild metric. Provided $M \neq 0$, one could set $M = 1$ by scaling, but we prefer to keep M in our formulas. We let g_S, g_K denote the Schwarzschild, respectively Kerr, metric, and $\Box_S$, $\Box_K$ denote the associated d'Alembertians, where the (scalar) d'Alembertian is given by $\Box = \nabla^\gamma\partial_\gamma$ with ∇ denoting the metric connection.

The Kerr metric has a singularity at $r = 0$ on the equator $\theta = \pi/2$. The apparent singularities at the roots of Δ, namely at the horizons $r = r_\pm := M \pm \sqrt{M^2 - a^2}$, are merely coordinate singularities. For a further discussion of the nature of $r_\pm$, which is not relevant for our results, we refer the reader to, e.g., [13],[25].

To remove the coordinate singularities at $r = r_\pm$, we may introduce Eddington-Finkelstein coordinates. See, e.g., [25]. To do so, we let r^*, v_+, and ϕ_+ solve

$$dr^* = (r^2+a^2)\Delta^{-1}dr, \qquad dv_+ = dt + dr^*, \qquad d\phi_+ = d\phi + a\Delta^{-1}dr.$$

The metric then takes the form

$$\begin{aligned} ds^2 = &-\Big(1-\frac{2Mr}{\rho^2}\Big)\, dv_+^2 + 2\, dr\, dv_+ - 4a\rho^{-2}Mr\sin^2\theta\, dv_+\, d\phi_+ - 2a\sin^2\theta\, dr\, d\phi_+ \\ &+ \rho^2 d\theta^2 + \rho^{-2}\Big[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\Big]\sin^2\theta\, d\phi_+^2, \end{aligned}$$

which is nondegenerate up to the metric singularity at $\rho = 0$.

For our purposes, the Boyer-Lindquist coordinates are convenient at spatial infinity but not near the event horizon, while the Eddington-Finkelstein coordinates are convenient near the event horizon but not at spatial infinity. To combine the two we replace the (t,ϕ) coordinates with $(\tilde v, \phi_+)$, as in [35] and [55], by defining

$$\tilde v = v_+ - \mu(r)$$

where μ is a smooth function of r. In these $(\tilde v, r, \phi_+, \theta)$ coordinates the metric has the form

$$\begin{aligned} ds^2 = &\Big(1-\frac{2Mr}{\rho^2}\Big)\, d\tilde v^2 + 2\left(1 - \Big(1-\frac{2Mr}{\rho^2}\Big)\mu'(r)\right) d\tilde v\, dr - 4a\rho^{-2}Mr\sin^2\theta\, d\tilde v\, d\phi_+ \\ &+ \left(2\mu'(r) - \Big(1-\frac{2Mr}{\rho^2}\Big)(\mu'(r))^2\right) dr^2 - 2a\theta\Big(1 + 2\rho^{-2}Mr\mu'(r)\Big)\sin^2\theta\, dr\, d\phi_+ + \rho^2\, d\theta^2 \\ &+ \rho^{-2}\Big[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\Big]\sin^2\theta\, d\phi_+^2. \end{aligned}$$

The function μ is selected to satisfy:

(i) $\mu(r) \geq r^*$ for $r > 2M$, with equality for $r > 5M/2$.

(ii) The surfaces $\tilde{v} = const$ are space-like, i.e.

$$\mu'(r) > 0, \qquad 2 - (1 - \frac{2Mr}{\rho^2})\mu'(r) > 0.$$

For r_e fixed satisfying $r_- < r_e < r_+$, we shall consider the wave equation

$$\Box_K u = F, \qquad u|_{\Sigma^-} = f, \qquad \tilde{T}u|_{\Sigma^-} = g \ , \tag{2.1}$$

in $\mathcal{M} = \{\tilde{v} \geq 0, \ r \geq r_e\}$ and with initial data on the space like surface $\Sigma^- = \mathcal{M} \cap \{\tilde{v} = 0\}$. The choice of r_e is unimportant, and for convenience we may simply use $r_e = M$ for all Kerr metrics with $a/M \ll 1$.

We use $\nabla\!\!\!/$ to denote the angular derivatives $\nabla\!\!\!/_i = \partial_i - \frac{x^i}{r}\partial_r$, where $x = r\omega$ is understood. We set

$$E[u](\Sigma^-) = \int_{\Sigma^-} \left(|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla\!\!\!/ u|^2\right) r^2 \, dr \, d\omega$$

to be the initial energy. More generally, we use

$$E[u](\tilde{v}_0) = \int_{\mathcal{M} \cap \{\tilde{v} = \tilde{v}_0\}} \left(|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla\!\!\!/ u|^2\right) r^2 \, dr \, d\omega$$

to denote the energy on the space-like slice $\tilde{v} = constant$. In particular, $E[u](\Sigma^-) = E[u](0)$.

In Minkowski space, the localized energy estimate for the wave equation states that

$$\|\partial u\|_{l_\infty^{-1/2}(L_t^2 L_x^2) \cap L_t^\infty L_x^2} + \|u\|_{l_\infty^{-3/2}(L_t^2 L_x^2)} \lesssim \|\partial u(0, \cdot)\|_{L_x^2} + \|\Box u\|_{l_1^{1/2}(L_t^2 L_x^2) + L_t^1 L_x^2}.$$

Such estimates first appeared in [40, 41, 42] and subsequently in, e.g., [52], [32], [47], [30, 31], [11], [36, 37], [51], and [28]. Such estimates are known to be fairly robust and variants were proved in [1], [36, 37], [38], [10], [9], and [50] for various nontrapping metric perturbations. The most basic proof of the estimate above involves integrating $\Box u$ against $f(r)\partial_r u + \frac{n-1}{2}\frac{f(r)}{r}u$, where $f(r) = r/(r + 2^j)$ and integrating by parts.

The above proofs implicitly rely heavily on the fact that all null geodesics escape to infinity. In the case that there are trapped rays, it is known [44] that a loss is necessary in order to have a variant of the localized energy estimate.

Members of the Kerr family of black holes contain trapped rays. This is easiest to describe in the Schwarzschild case where trapping occurs on the event horizon $r = 2M$ and on the so-called photon sphere $r = 3M$. Utilizing the red shift effect as in [16] renders the trapping at $r = 2M$ inconsequential. The known localized energy estimates on the Schwarzschild space-time [6, 7], [8], [15, 16], and [35] reflect a loss at the photon sphere. These can be proved by choosing a related multiplier where the f is more complicated and switches sign at the photon sphere.

The trapping on the Kerr space-times is more delicate and can only be described in phase-space, though it does occur within an $O(a)$ neighborhood of $r = 3M$. See, e.g., [55]. Since the region containing trapped rays cannot be described only in physical space, it is provable [2] that no first order differential multipler, as used above, can yield such a localized energy estimate. Despite this, there have been three related but distinct approaches that have yielded localized energy estimates on Kerr backgrounds with small angular momentum. See [3], [17], and [55]. See, also, [18, 19] for the subextremal case $|a| < M$.

The approach that we shall follow is that of [55]. We define our localized energy norm as

$$\|u\|_{LE} = \|(1-\tilde\chi)\nabla u\|_{l_\infty^{-1/2}(L^2_{\tilde v,r,\omega})} + \|u\|_{l_\infty^{-3/2}(L^2_{\tilde v,r,\omega})} \tag{2.2}$$

where $\tilde\chi = \tilde\chi(r)$ is a smooth radial cut off function supported in $[2.5M, 3.5M]$ and is identity in a neighborhood containing all of the trapped rays. It is this degeneracy of the norm that represents the loss due to the trapping.

We then have the following

Lemma 2.1. *Suppose $a \ll M$. Let u solves the inhomogeneous wave equation $\Box_K u = F_1 + F_2$ in the region $\mathcal{M}$ where F_2 is supported in $\{r \geq 3.5M\}$. Then we have*

$$\sup_{\tilde v\geq 0} E[u](\tilde v) + \|u\|^2_{LE} \lesssim E[u](\Sigma^-) + \|F_1\|^2_{L^1_{\tilde v}L^2_{r,\omega}} + \|F_2\|^2_{l_1^{1/2}(L^2_{\tilde v,r,\omega})}. \tag{2.3}$$

This is an easy corollary of [55, Theorem 4.1]. Indeed, the norm $\|\cdot\|_{LE^1_K}$ defined therein satisfies

$$\|u\|_{LE} \lesssim \|u\|_{LE^1_K}$$

and, under the support conditions on F_2,

$$\|F_2\|_{l_1^{1/2}(L^2_{\tilde v,r,\omega})} \gtrsim \|F_2\|_{LE^*_K},$$

where the latter is defined in [55]. Moreover, the control of F_1 follows by bootstrapping [55, (4.14)] using the energy term rather than the localized energy norm.

We also have the following higher order version of (2.3), which similarly follows from [55, Theorem 4.5].

Corollary 2.2. *Let n be a positive integer, and suppose that u solves the inhomogeneous wave equation $\Box_K u = F_1 + F_2$ in the region $\mathcal{M}$ where F_2 is supported in $\{r \geq 3.5M\}$. Then we have*

$$\begin{aligned}\sup_{\tilde v\geq 0}\sum_{|\alpha|\leq n} E[\partial^\alpha u](\tilde v) &+ \sum_{|\alpha|\leq n}\|\partial^\alpha u\|^2_{LE} \\ &\lesssim \|u(0,\cdot)\|^2_{H^{n+1}_x} + \|\tilde T u(0,\cdot)\|^2_{H^n_x} + \sum_{|\alpha|\leq n}\|\partial^\alpha F_1\|^2_{L^1_{\tilde v}L^2_{r,\omega}} + \sum_{|\alpha|\leq n}\|\partial^\alpha F_2\|^2_{l_1^{1/2}(L^2_{\tilde v,r,\omega})}.\end{aligned} \tag{2.4}$$

3. Weighted Strichartz estimates

In this section, we collect the required Sobolev-type and weighted Strichartz estimates.

3.1. Weighted Sobolev estimates.

In the sequel, we shall require the following weighted Sobolev estimates. These are straightforward variants of those that appeared in, e.g., [33].

Lemma 3.1. *For $R \geq 10$, $2 \leq p < \infty$, and any $b \in \mathbb{R}$, we have*

$$\|r^b v\|_{L_r^{\frac{2p(p-1)}{p-2}} L^\infty_\omega(r\geq R+1)} \lesssim \sum_{|\gamma|\leq 2}\|r^{b-\frac{1}{p-1}} Y^\gamma v\|_{L^p_r L^2_\omega(r\geq R)}, \tag{3.1}$$

$$\|r^b v\|_{L^4_x(|x|\geq R+1)} \lesssim \sum_{|\gamma|\leq 1} \|r^{b+\frac{1}{2}-\frac{2}{p}} Y^\gamma v\|_{L^p_r L^2_\omega(r\geq R)}, \quad p \leq 4, \tag{3.2}$$

and

$$\|r^b v\|_{L^\infty_x(|x|\geq R+1)} \lesssim \sum_{|\gamma|\leq 2} \|r^{b-\frac{2}{p}} Y^\gamma v\|_{L^p_r L^2_\omega(r\geq R)}. \tag{3.3}$$

Proof. By Sobolev's lemma on $\mathbb{R}\times\mathbb{S}^2$, we have for each $j\in\mathbb{N}$ the uniform bounds

$$\|v\|_{L^\infty_r L^\infty_\omega([j,j+1]\times S^2)} \lesssim \sum_{|\gamma|\leq 2} \Big(\int_{j-1}^{j+2}\int_{S^2} |Y^\gamma v|^2\, d\omega dr\Big)^{\frac{1}{2}}.$$

Hence,

$$\|v\|_{L^\infty_r([j,j+1])L^\infty_\omega} \lesssim j^{-1} \sum_{|\gamma|\leq 2} \|Y^\gamma v\|_{L^2_r([j-1,j+2])L^2_\omega}. \tag{3.4}$$

Or more generally,

$$\|r^b v\|_{L^\infty_r([j,j+1])L^\infty_\omega} \lesssim \sum_{|\gamma|\leq 2} \|r^{b-1} Y^\gamma v\|_{L^2_r([j-1,j+2])L^2_\omega}.$$

The factor j^{-1} on the right comes from the fact that the volume element for $\mathbb{R}^3$ is $r^2 dr d\omega$. By Hölder's inequality, we have that for every $1\leq q<\infty$ and $p>2$

$$\|r^b v\|_{L^q_r([j,j+1])L^\infty_\omega} \lesssim \sum_{|\gamma|\leq 2} \|r^{b+\frac{2}{q}-\frac{2}{p}} Y^\gamma v\|_{L^p_r([j-1,j+2])L^2_\omega}. \tag{3.5}$$

This is just the inequality (3.1) if we set $q=\frac{2p(p-1)}{p-2}$ and l^p-sum over $j\geq R+1$ using the Minkowski integral inequality. Estimate (3.3) follows from obvious modifications of the same argument.

Inequality (3.2) follows from a similar argument. The proof of (3.4) also yields

$$\|v\|_{L^4_r([j,j+1])L^4_\omega} \lesssim j^{-\frac{1}{2}} \sum_{|\gamma|\leq 1} \|Y^\gamma v\|_{L^2_r([j-1,j+2])L^2_\omega},$$

which implies (3.2) after an application of Hölder's inequality, weighting appropriately, and l^4-summing over j. ∎

3.2. **Weighted Strichartz estimates.** In this subsection, we prove inhomogeneous weighted Strichartz estimates near spatial infinity. The exact form of the Kerr metric is not important here; all that matters is that it is a small perturbation of Minkowski. We shall first prove an estimate for small perturbations of the Minkowski space-time. In the sequel, we shall then proceed to cutoff the Kerr solution and focus on the exterior of a ball of sufficiently large radius that we may view the Kerr metric as a small asymptotic perturbation of the Minkowski metric.

To this end, we set

$$\Box_h \phi = (\partial_t^2 - \Delta - \partial_\alpha h^{\alpha\beta}(t,x)\partial_\beta)\phi,$$

where the summation convention is employed. We shall assume that

$$h^{\alpha\beta}=h^{\beta\alpha}, \quad |h|\leq \frac{\delta}{\langle x\rangle^\rho}, \quad |\partial h|\leq \frac{\delta}{\langle x\rangle^{1+\rho}}, \tag{3.6}$$

for some $\rho > 0$ and $\delta \ll 1$, where in an abuse of notation we set

$$|h| = \sum_{\alpha,\beta=0}^{3} |h^{\alpha\beta}(t,x)|, \quad |\partial h| = \sum_{\alpha,\beta,\gamma=0}^{3} |\partial_\gamma h^{\alpha\beta}(t,x)|.$$

Our main estimate near infinity is the following weighted Strichartz estimate, which is closely akin to those first proved in [27] and [21].

Theorem 3.2. *Let $p \in [2,\infty)$. Suppose w solves*

$$\Box_h w = G_1 + G_2, \qquad w(0,\cdot) = 0 = \partial_t w(0,\cdot),$$

where h satisfies (3.6) for δ sufficiently small. Additionally, suppose that w vanishes in a neighborhood of the origin. Then for any $\delta_1 > 0$ and $1/2 - 1/p < s < 1/2$ we have

$$\|\langle r\rangle^{\frac{3}{2}-\frac{4}{p}-s} w\|_{L^p_{t,r}L^2_\omega} \lesssim \|r^{-\frac{1}{2}-s} G_1\|_{L^1_{t,r}L^2_\omega} + \|\langle r\rangle^{\frac{3}{2}-s+\delta_1} G_2\|_{L^2_{t,r,\omega}}. \tag{3.7}$$

We will prove this by interpolating between the estimates that the following two lemmas afford to us. The first is a standard localized energy estimate, while the second result is a variant of such where we have divided through by a derivative in the spirit of, e.g., [58, Lemma 2.3].

To begin, we note that the following localized energy estimate is an immediate corollary of the methods of [36]. See also [37] and [38].

Lemma 3.3. *Suppose that $h^{\alpha\beta}$ are smooth and satisfy (3.6) for $\delta \ll 1$ sufficiently small. Let $G \in L^1_t L^2_x + l^{1/2}(L^2_t L^2_x)$, and let w solve $\Box_h w = G$ on $\mathbb{R}_+ \times \mathbb{R}^3$. Then*

$$\|\partial w\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty (L^2_t L^2_x)} + \|w\|_{l^{-3/2}_\infty (L^2_t L^2_x)} \lesssim \|\partial w(0,\,\cdot\,)\|_2 + \|G\|_{L^1_t L^2_x + l^{1/2}_1 (L^2_t L^2_x)}. \tag{3.8}$$

The other endpoint for our real interpolation shall be:

Lemma 3.4. *Suppose that $G \in l^{1/2}_1(L^2_t \dot H^{-1}_x) + L^1_t \dot H^{-1}_x$ and that w solves $\Box_h w = G$ with vanishing initial data. Here h is assumed to satisfy (3.6) for $\delta \ll 1$ sufficiently small. Then*

$$\|w\|_{l^{-1/2}_\infty (L^2_t L^2_x) \cap L^\infty_t L^2_x} \lesssim \|G\|_{l^{1/2}_1 (L^2_t \dot H^{-1}_x) + L^1_t \dot H^{-1}_x}. \tag{3.9}$$

Proof. Given a function $F \in l^{1/2}_1(L^2_t L^2_x) + L^1_t L^2_x$ over a $[0,T]$ time-strip, let u solve $\Box_h u = F$ with vanishing data on the $t = T$ slice. Applying Lemma 3.3 backward in time, we obtain

$$\|u\|_{l^{-1/2}_\infty (L^2_t \dot H^1_x) \cap L^\infty_t \dot H^1_x} \lesssim \|F\|_{l^{1/2}_1 (L^2_t L^2_x) + L^1_t L^2_x}. \tag{3.10}$$

We then have

$$\langle w, F\rangle = \langle w, \Box_h u\rangle = \langle \Box_h w, u\rangle \le \|\Box_h w\|_{l^{1/2}_1 (L^2_t \dot H^{-1}_x) + L^1_t \dot H^{-1}_x} \|u\|_{l^{-1/2}_\infty (L^2_t \dot H^1_x) \cap L^\infty_t \dot H^1_x},$$

where the inner product is that from $L^2_{t,x}([0,T]\times\mathbb{R}^3)$. Applying (3.10), we get

$$\langle w, F\rangle \lesssim \|G\|_{l^{1/2}_1 (L^2_t \dot H^{-1}_x) + L^1_t \dot H^{-1}_x} \|F\|_{l^{1/2}_1 (L^2_t L^2_x) + L^1_t L^2_x},$$

which by duality completes the proof. ∎

We can now use real interpolation to complete the proof of Theorem 3.2. We record the following facts:

- ([5, Theorem 5.6.1]) $[l^{s_0}_{q_0}(A), l^{s_1}_{q_1}(A)]_{\theta,q} = l^{(1-\theta)s_0+\theta s_1}_q(A)$ for any $s_0 \neq s_1$, $0 < q_0, q_1, q \le \infty$ and $\theta \in (0,1)$.
- ([5, Theorem 5.6.2]) $[l^{s_0}_{q_0}(A_0), l^{s_1}_{q_1}(A_1)]_{\theta,q} = l^{(1-\theta)s_0+\theta s_1}_q([A_0, A_1]_{\theta,q})$ if $0 < q_0, q_1, q < \infty$ and $\frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}$.
- ([56, 1.18.4]) $[L^{p_0}(A_0), L^{p_1}(A_1)]_{\theta,p} = L^p([A_0, A_1]_{\theta,p})$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 \le p_0, p_1 < \infty$.
- ([5, Theorem 6.4.5]) $[\dot H^{s_0}, \dot H^{s_1}]_{\theta,2} = \dot H^{(1-\theta)s_0+\theta s_1}$.

We note that the left side of (3.8) controls

$$\|w\|_{l^{-1/2}_\infty(L^2_t \dot H^1_x)\cap l^{-3/2}_\infty(L^2_t L^2_x)} + \|w\|_{L^\infty_t \dot H^1_x}.$$

Thus, interpolation between (3.8) and (3.9) yields

$$\|w\|_{L^\infty_t \dot H^{s'}_x} + \|w\|_{l_2^{-1/2-s'}(L^2_t L^2_x)} \lesssim \|G_1\|_{L^1_t \dot H^{s'-1}_x} + \|G_2\|_{l_1^{1/2}(L^2_t \dot H^{s'-1}_x)} \tag{3.11}$$

for any $0 < s' < 1$. (Technically, for $i = 1, 2$, we are letting w_i solve $\Box_h w_i = G_i$ with vanishing initial data and doing separate interpolations for each i. We record this, however, as a single step.) Combining the above with the trace theorem on the sphere,

$$\|r^{\frac{3}{2}-s''} f\|_{L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot H^{s''}}, \quad 1/2 < s'' < 3/2, \tag{3.12}$$

we obtain

$$\|r^{\frac{3}{2}-s''} w\|_{L^\infty_{t,r} L^2_\omega} \lesssim \|G_1\|_{L^1_t \dot H^{s''-1}_x} + \|G_2\|_{l_1^{1/2}(L^2_t \dot H^{s''-1}_x)}, \quad 1/2 < s'' < 1. \tag{3.13}$$

Interpolating between the second term in (3.11) and (3.13) then yields

$$\|r^{\frac{3}{2}-\frac{4}{p}-s} w\|_{L^p_{t,r} L^2_\omega} \lesssim \|G_1\|_{L^1_t \dot H^{s-1}_x} + \|G_2\|_{l_1^{1/2}(L^2_t \dot H^{s-1}_x)}, \quad \frac{1}{2} - \frac{1}{p} < s < 1. \tag{3.14}$$

If additionally $1 - s > 1/2$, then we can apply the dual to (3.12) to obtain

$$\|r^{\frac{3}{2}-\frac{4}{p}-s} w\|_{L^p_{t,r} L^2_\omega} \lesssim \|r^{-\frac{1}{2}-s} G_1\|_{L^1_{t,r} L^2_\omega} + \|G_2\|_{l_1^{1/2}(L^2_t \dot H^{s-1}_x)}, \quad \frac{1}{2} - \frac{1}{p} < s < \frac{1}{2}. \tag{3.15}$$

By duality, the Sobolev embedding $\dot H^{1-s} \subset L^{\frac{6}{2s+1}}$, and Hölder's inequality, we have

$$\|G_2\|_{l_1^{1/2}(L^2_t \dot H^{s-1}_x)} \lesssim \|\langle r\rangle^{\frac{3}{2}-s+\delta} G_2\|_{L^2_{t,r,\omega}}, \quad s \in (0, 1/2), \tag{3.16}$$

which completes the proof of Theorem 3.2.

4. The Strauss conjecture on Kerr black hole background

We now prove our main theorem, Theorem 1.1. In fact, with more notation in place, we first state a more precise version of the theorem.

Theorem 4.1. *Suppose that the initial data $(f, g) \in H^3 \times H^2$ have compact support. Then there exists a global solution u in $\mathcal{M}$ for the problem (1.1) with $p > 1 + \sqrt{2}$, provided that*

$$\|f\|_{H^3} + \|g\|_{H^2} = \varepsilon \ll 1 \tag{4.1}$$

is small enough. Moreover, there is a large constant R_0, depending only on M and a, such that we have the following property: For any initial data supported in a ball with *radius R with $R \geq R_0$ and any $\delta > 0$, there exists a constant $C > 0$, depending on F_p, R, δ, M and a, so that we have the following estimate for the solution u,*

$$\sum_{|\gamma|\leq 2} \Big(\|r^{-\alpha}\chi Z^\gamma u\|_{L^q_{\tilde v} L^q_r L^2_\omega} + \|\langle r\rangle^{-3/2-\delta}\partial^\gamma u\|_{L^2_{\tilde v} L^2_r L^2_\omega} + \|\partial^\gamma \partial u\|_{L^\infty_{\tilde v} L^2_r L^2_\omega}\Big) \leq C\varepsilon.$$

Here, $\chi(r)$ is a cutoff function supported when $r > R$ so that $\chi = 1$ when $r > R+1$, and $\alpha = \frac{4}{q} - \frac{2}{q-1}$ with $q = p$ if $p \in (1+\sqrt{2}, 3)$ and $q \in (1+\sqrt{2}, 3)$ if $p \geq 3$.

In the proof that follows, we shall only focus on $p \in (1+\sqrt{2}, 3)$. This is, in part, because the cases $p > 3$ have been handled in previous work. The adjustments to our proof needed to explore the cases $p \geq 3$ are straightforward. Indeed, you simply iterate in the corresponding spaces for any index q in the $(1+\sqrt{2}, 3)$ range and use Sobolev embeddings to bound the $p-q$ extra copies of the solution in the nonlinearity.

4.1. **Setting.** We are interested in solving the equation (1.1)

$$\Box_K u = F_p(u), \qquad u|_{\Sigma^-} = f, \qquad \tilde{T}u|_{\Sigma^-} = g, \tag{4.2}$$

where we assume (1.2). The initial data are taken to have compact support and to be subject to (4.1). We choose $R > 3.5M$ sufficiently large so that the supports of f and g are contained within $\{r \leq R\}$ and so that $(\rho^2/r^2)\Box_K$ satisfies (3.6) on $\{r \geq R\}$.

We let $\chi \in C^\infty(\mathbb{R})$ satisfy $0 \leq \chi(r) \leq 1$, $\chi(r) \equiv 0$ for $r \leq R$, and $\chi(r) \equiv 1$ for $r > R+1$. We shall utilize Theorem 3.2 with $s = \frac{3}{2} - \frac{2}{p-1}$, which falls in the range $(1/2 - 1/p, 1/2)$ precisely when $1+\sqrt{2} < p < 3$. For $\alpha = \frac{4}{p} - \frac{2}{p-1} = \frac{2(p-2)}{p(p-1)}$, we define

$$\|\phi\|_X = \sum_{|\gamma|\leq 2} \Big(\|r^{-\alpha}\chi Z^\gamma \phi\|_{L^pL^pL^2} + \|\langle r\rangle^{-3/2-\delta}\partial^\gamma\phi\|_{L^2L^2L^2} + \|\partial^\gamma\partial\phi\|_{L^\infty L^2L^2}\Big), \tag{4.3}$$

$$\|g\|_N = \sum_{|\gamma|\leq 2} \Big(\|r^{-\alpha p}\chi^p Z^\gamma g\|_{L^1L^1L^2} + \|Z^\gamma g\|_{L^1L^2L^2}\Big) . \tag{4.4}$$

4.2. **Main estimate.** In this section, we shall combine the weighted Strichartz estimates, which are known to hold for small perturbations, and the localized energy estimates, which are known to hold on Kerr provided $a \ll M$, to prove our main estimate. We shall also incorporate the necessary vector fields and, as such, shall examine the associated commutators.

Lemma 4.2. *Let u be the solution to*

$$\Box_K u = G, \qquad u|_{\Sigma^-} = f, \qquad \tilde{T}u|_{\Sigma^-} = g. \tag{4.5}$$

where f, g, $G(0, r\omega)$, and $\partial_{\tilde v} G(0, r\omega)$ are supported in the region $\{r \leq R\}$. Then there exists a constant R_0, such that for $R > R_0$, we have

$$\|u\|_X \lesssim \|f\|_{H^3} + \|g\|_{H^2} + \|G\|_N \ . \tag{4.6}$$

Proof. We first note that the latter two terms of (4.3) are trivially controlled by the right side of (4.6) using (2.4).

Next, we record that

$$[\Box_K, \partial]u = \mathcal{O}(r^{-2}|\partial^2 u| + r^{-3}|\partial u|)$$

and

$$[\Box_K, \Omega]u = \mathcal{O}(ar^{-2}|\partial^2 u| + ar^{-3}|\partial u|).$$

More generally,

$$[\Box_K, \Omega^\alpha \partial^\beta]u = \mathcal{O}(r^{-2}) \sum_{\substack{|\mu|+|\nu|\le|\alpha|+|\beta| \\ |\mu|\le|\alpha|-1}} |\Omega^\mu \partial^\nu \partial u| + \mathcal{O}(r^{-3}) \sum_{\substack{|\mu|+|\nu|\le|\alpha|+|\beta| \\ |\mu|\le|\alpha|-1}} |\Omega^\mu \partial^\nu u|. \tag{4.7}$$

Thus, we see that if $|\alpha| = 1$, using (2.4) with $n = |\beta|$,

$$\begin{aligned} \|\partial^\beta \chi \Omega^\alpha u\|_{LE} &\lesssim \|\Omega^\alpha f\|_{H^{|\beta|+1}} + \|\Omega^\alpha g\|_{H^{|\beta|}} \\ &+ \sum_{|\nu|\le|\beta|} \|\partial^\nu \Omega^\alpha G\|_{L^1L^2L^2} + \sum_{|\nu|\le|\beta|} \|\partial^\gamma [\Box_K, \chi]\Omega^\alpha u\|_{l_1^{1/2}(L^2L^2L^2)} \\ &+ \sum_{|\nu|\le|\beta|} \|\partial^\nu \chi [\Box_K, \Omega^\alpha]u\|_{l_1^{1/2}(L^2L^2L^2)}. \end{aligned} \tag{4.8}$$

As $[\Box_K, \chi]$ is compactly supported and the coefficients of Ω are $\mathcal{O}(1)$ on the support of $[\Box_K, \chi]$, we have

$$\sum_{|\nu|\le|\beta|} \|\partial^\gamma [\Box_K, \chi]\Omega^\alpha u\|_{l_1^{1/2}(L^2L^2L^2)} \lesssim \sum_{|\nu|\le|\beta|+1} \|\partial^\nu u\|_{LE}.$$

Similarly, using the commutator estimate above,

$$\sum_{|\nu|\le|\beta|} \|\partial^\nu \chi [\Box_K, \Omega^\alpha]u\|_{l_1^{1/2}(L^2L^2L^2)} \lesssim \sum_{|\nu|\le|\beta|+1} \|\partial^\nu u\|_{LE}.$$

If we, in turn, apply (2.4) with $n = |\beta| + 1$, it follows that

$$\sum_{\substack{|\alpha|+|\beta|\le 2 \\ |\alpha|\le 1}} \|\partial^\beta \chi Z^\alpha u\|_{LE} \lesssim \|f\|_{H^3} + \|g\|_{H^2} + \sum_{|\alpha|\le 2} \|Z^\alpha G\|_{L^1L^2L^2}. \tag{4.9}$$

Higher order estimates akin to this have previously appeared in, e.g., [39].

We now turn to bounding the first term in (4.3). We note that

$$\Box_K \chi Z^\gamma u = \chi Z^\gamma G + [\Box_K, \chi]Z^\gamma u + \chi[\Box_K, Z^\gamma]u.$$

We also note that $(\rho^2/r^2)\Box_K$ satisfies the requirements (3.6) on the support of χ when R is sufficiently large. The support conditions on f, g, and G guarantee that the Cauchy data for $\chi Z^\gamma u$ vanish. Using that ρ^2/r^2 is $\mathcal{O}(1)$ on the support of χ, it follows from (3.7) that

$$\begin{aligned} \sum_{|\gamma|\le 2} \|r^{-\alpha} \chi Z^\gamma u\|_{L^pL^pL^2} &\lesssim \sum_{|\gamma|\le 2} \|r^{-\alpha p} \chi Z^\gamma G\|_{L^1L^1L^2} \\ &+ \sum_{|\gamma|\le 2} \|r^{\frac{3}{2}-s+\delta} [\Box_K, \chi] Z^\gamma u\|_{L^2L^2L^2} + \sum_{|\gamma|\le 2} \|r^{\frac{3}{2}-s+\delta} \chi [\Box_K, Z^\gamma]u\|_{L^2L^2L^2}. \end{aligned} \tag{4.10}$$

We first note that

$$\|r^{-\alpha p} \chi Z^\gamma G\|_{L^1L^1L^2} \lesssim \|r^{-\alpha p} \chi^p Z^\gamma G\|_{L^1L^1L^2} + \|Z^\gamma G\|_{L^1L^2L^2}$$

since $\chi - \chi^p$ is supported where $r \in [R, R+1]$. As the weight in the second term in the right side of (4.10) is $\mathcal{O}(1)$ on the support of $[\Box_K, \chi]$ and as that support is contained in

$\{r \geq 3.5M\}$, it follows that the second term on the right is bounded by $\sum_{\gamma\le 2} \|\partial^\gamma u\|_{LE}$ and can be controlled using (2.4). Similarly, we can choose $0 < \delta < s$ and use (4.7) to control the last term on the right by

$$\sum_{\substack{|\gamma|+|\mu|\le 2\\ |\gamma|\le 1}} \|\partial^\mu \chi Z^\gamma u\|_{LE} + \sum_{|\mu|\le 2} \|\partial^\mu u\|_{LE},$$

for which (4.9) and (2.4) provide the desired bound. □

4.3. **The Strauss conjecture.** We can now prove Theorem 4.1. We solve (4.2) via iteration. We set $u_0 \equiv 0$ and recursively define u_{k+1} to be the solution to the linear equation

$$\Box_K u_{k+1} = F_p(u_k), \qquad u|_{\Sigma^-} = f, \qquad \tilde{T}u|_{\Sigma^-} = g. \tag{4.11}$$

Boundedness: By the smallness condition (4.1) on the data as well as the condition imposed on their supports, it follows from Lemma 4.2 that there is a universal constant C_1 so that

$$\|u_1\|_X \le C_1\varepsilon$$

and

$$\|u_{k+1}\|_X \le C_1\varepsilon + C_1\|F_p(u_k)\|_N.$$

We shall argue inductively to prove that

$$\|u_{k+1}\|_X \le 2C_1\varepsilon.$$

By the above, it suffices to show

$$\|F_p(u_k)\|_N \le \varepsilon.$$

By condition (1.2), we have

$$|Z^\gamma F_p(u_k)| \lesssim |u_k|^{p-1}|Z^\gamma u_k| + |u_k|^{p-2}\Big(\sum_{|\beta|\le 1} |Z^\beta u_k|\Big)^2$$

for $|\gamma| \le 2$.

We start with bounding the first term in (4.4). We first note that

$$\sum_{|\gamma|\le 2} \|r^{-\alpha p}\chi^p Z^\gamma F_p(u_k)\|_{L^1L^1L^2} \lesssim \|r^{-\alpha}\chi u_k\|^{p-1}_{L^pL^pL^\infty} \sum_{|\gamma|\le 2} \|r^{-\alpha}\chi Z^\gamma u_k\|_{L^pL^pL^2}$$
$$+ \|r^{-\alpha}\chi u_k\|^{p-2}_{L^pL^pL^\infty}\Big(\sum_{|\beta|\le 1} \|r^{-\alpha}\chi Z^\beta u_k\|_{L^pL^pL^4}\Big)^2.$$

By the $H^2_\omega \subset L^\infty_\omega$ and $H^1_\omega \subset L^4_\omega$ Sobolev embeddings on $\mathbb{S}^2$, it follows that the right side above is $\mathcal{O}(\|u_k\|^p_X)$.

We now proceed to the second term in (4.4). We first observe that

$$\sum_{|\gamma|\le 2} \|u_k^{p-1} Z^\gamma u_k\|_{L^1L^2L^2(r\ge R+2)} \lesssim \|r^{\frac{\alpha}{p-1}} u_k\|^{p-1}_{L^pL^{\frac{2p(p-1)}{p-2}}L^\infty(r\ge R+2)} \sum_{|\gamma|\le 2} \|r^{-\alpha}\chi Z^\gamma u_k\|_{L^pL^pL^2}$$

Applying (3.1), it follows that the right side is

$$\lesssim \Big(\sum_{|\gamma|\le 2} \|r^{\frac{\alpha-1}{p-1}}\chi Z^\gamma u_k\|_{L^pL^pL^2}\Big)^{p-1} \sum_{|\gamma|\le 2} \|r^{-\alpha}\chi Z^\gamma u_k\|_{L^pL^pL^2}.$$

As $\frac{\alpha-1}{p-1} \le -\alpha$ for $p\le 3$, it follows that this is also $\mathcal{O}(\|u_k\|_X^p)$. Moreover,

$$\sum_{|\gamma|\le 2} \|u_k^{p-1} Z^\gamma u_k\|_{L^1L^2L^2(r\le R+2)} \lesssim \|u_k\|^{p-2}_{L^\infty L^\infty L^\infty(r\le R+2)} \|u_k\|_{L^2L^\infty L^\infty (r\le R+2)} \sum_{|\gamma|\le 2} \|\partial^\gamma u_k\|_{L^2L^2L^2(r\le R+2)}.$$

Sobolev embeddings allow us to control this by

$$\Big(\sum_{|\gamma|\le 2} \|\partial^\gamma \partial u_k\|_{L^\infty L^2L^2}\Big)^{p-2}\Big(\sum_{|\gamma|\le 2} \|\partial^\gamma u_k\|_{L^2L^2L^2(r\le R+3)}\Big)^2, \tag{4.12}$$

which is also $\mathcal{O}(\|u_k\|_X^p)$.

We similarly examine

$$\Big\|u_k^{p-2}\sum_{|\beta|\le 1} Z^\beta u_k\Big\|_{L^1L^2L^2(r\ge R+2)} \lesssim \|r^{\frac{1}{p}} u_k\|^{p-2}_{L^pL^\infty L^\infty(r\ge R+2)} \Big(\sum_{|\beta|\le 1} \|r^{-\frac{1}{p}+\frac{4-p}{2p}} Z^\beta u_k\|_{L^pL^4L^4(r\ge R+2)}\Big)^2.$$

Applications of (3.2) and (3.3), it follows that this is bounded by

$$\Big(\sum_{|\gamma|\le 2} \|r^{-\frac{1}{p}}\chi Z^\gamma u_k\|_{L^pL^pL^2}\Big)^p.$$

As $\alpha \le 1/p$ for $p\le 3$, we have that these terms are $\mathcal{O}(\|u_k\|_X^p)$. It remains to bound

$$\Big\|u_k^{p-2}\sum_{|\beta|\le 1} Z^\beta u_k\Big\|_{L^1L^2L^2(r\le R+2)} \lesssim \|u_k\|^{p-2}_{L^\infty L^\infty L^\infty(r\le R+2)}\Big(\sum_{|\beta|\le 1}\|Z^\beta u_k\|_{L^2L^4L^4(r\le R+2)}\Big)^2.$$

Using Sobolev embeddings, this is estimated by (4.12), which as noted above is $\mathcal{O}(\|u_k\|_X^p)$.

Combining the above, we have

$$\|F_p(u_k)\|_N \le C\|u_k\|_X^p \le C(C_1\varepsilon)^p,$$

where we have employed the inductive hypothesis. As long as ε is chosen sufficiently small that $CC_1^p\varepsilon^{p-1}\le 1$, the proof of boundedness is complete.

Convergence of the sequence $\{u_k\}$*:* We shall complete the proof by showing that the sequence $\{u_k\}$ is Cauchy in X. For $k\ge 1$, we have

$$\|u_{k+1}-u_k\|_X \le C_1\|F_p(u_k)-F_p(u_{k-1})\|_N.$$

Mimicking the proof above shows that

$$\begin{aligned}\|F_p(u_k)-F_p(u_{k-1})\|_N &\le C(\|u_k\|_X^{p-1}+\|u_{k-1}\|_X^{p-1})\|u_k-u_{k-1}\|_X\\ &\le 2CC_1^{p-1}\varepsilon^{p-1}\|u_k-u_{k-1}\|_X.\end{aligned}$$

For all ε sufficiently small, this implies that

$$\|u_{k+1} - u_k\|_X \le \frac{1}{2}\|u_k - u_{k-1}\|_X,$$

which suffices to show that the sequence is Cauchy, and hence completes the proof of Theorem 4.1.

5. The Strauss conjecture on the Schwarzschild background

As pointed out before, the technical assumption of compactly support for the initial data can be removed in the case of the Schwarzschild background ($a = 0$) by adapting the arguments in [50] and [58].

Consider the evolution of the nonlinear waves in the cylindrical region $\mathcal{M}$,

$$\Box_S u = F_p(u), \qquad u_{|\Sigma^-} = f, \qquad \tilde{T}u_{|\Sigma^-} = g\ . \tag{5.1}$$

We have the following theorem, which is analogous to Theorem4.1:

Theorem 5.1. *Let $p > 1 + \sqrt{2}$. Then there exists a global solution u in $\mathcal{M}$ for the problem (5.1), provided that the initial data (f, g) satisfy*

$$E[f,g] := \sum_{|\gamma|\le 3} \|Y^\gamma f\|_{\dot{H}^s \cap L^2} + \sum_{|\gamma|\le 2} \|Y^\gamma g\|_{\dot{H}^{s-1}\cap L^2} < \epsilon \ll 1$$

for $s = s(p) = \frac{3}{2} - \frac{2}{q-1}$. Moreover, there is a large constant R_0, depending only on M, such that we have the following property: for any $\delta > 0$, there exists a $C > 0$, depending on F_p, $R \ge R_0$, δ and M, so that we have the following estimate for u:

$$\sum_{|\gamma|\le 2} \left(\|r^{-\alpha}\chi Y^\gamma u\|_{L^q_{\tilde{v}} L^q_r L^2_\omega} + \|\langle r\rangle^{-3/2-\delta}\nabla_x^\gamma u\|_{L^2_{\tilde{v}} L^2_r L^2_\omega} + \|\nabla_x^\gamma \partial u\|_{L^\infty_{\tilde{v}} L^2_r L^2_\omega} \right) \le C\varepsilon.$$

Here $q = p$ if $p \in (1+\sqrt{2}, 3)$ and $q \in (1+\sqrt{2}, 3)$ if $p \ge 3$, $\alpha = \frac{4}{q} - \frac{2}{q-1}$, and $\chi(r)$ is a cutoff function supported when $r > R$ so that $\chi = 1$ when $r > R + 1$.

Remark 5.1. *The key fact for us to prove Theorem 5.1 is that we can rewrite the D'Alembertian as $\partial_t^2 + P$ for a certain self-adjoint time-independent Laplacian P.*

The only important change in the argument is the following version of the weighted Strichartz estimates, which replaces Theorem 3.2

Proposition 5.2 (Weighted Strichartz estimates for Schwarzschild)**.** *Let u solve the equation $\Box_S u = G_1 + G_2$ with initial data (f, g) on $\{\tilde{v} = 0\}$. Moreover, assume that u vanishes in the region $\{r < KM\}$ for some large number $K > 0$. Then for any $p \ge 2$, $1/2 - 1/p < s < 1/2$ and $\delta > 0$, we have*

$$\begin{aligned} \|r^{3/2-4/p-s}u\|_{L^q_{\tilde{v},r}L^2_\omega(\mathcal{M})} \quad &\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} \\ &+\|r^{-1/2-s}F_1\|_{L^1_{\tilde{v},r}L^2_\omega} + \|r^{3/2-s+\delta}F_2\|_{L^2_{\tilde{v},x}}. \end{aligned} \tag{5.2}$$

Let us now prove the Strauss conjecture, Theorem 5.1, based on this Proposition.

We choose $R = KM$, where K is large enough so that Proposition 5.2 holds. We let $\chi \in C^\infty(\mathbb{R})$ satisfy $0 \le \chi(r) \le 1$, $\chi(r) \equiv 0$ for $r \le R$, and $\chi(r) \equiv 1$ for $r > R+1$. For $\alpha = \frac{4}{p} - \frac{2}{p-1}$, we define

$$\|\phi\|_X = \sum_{|\gamma|\le 2} \Big(\|r^{-\alpha}\chi Y^\gamma \phi\|_{L^pL^pL^2} + \|\langle r\rangle^{-3/2-\delta}\nabla_x^\gamma \phi\|_{L^2L^2L^2} + \|\nabla_x^\gamma \partial\phi\|_{L^\infty L^2L^2}\Big), \tag{5.3}$$

$$\|g\|_N = \sum_{|\gamma|\le 2} \Big(\|r^{-\alpha p}\chi^p Y^\gamma g\|_{L^1L^1L^2} + \|Y^\gamma g\|_{L^1L^2L^2}\Big) , \tag{5.4}$$

where $\delta > 0$ is arbitrarily fixed.

We can now utilize Proposition 5.2 with $s = \frac{3}{2} - \frac{2}{p-1}$, which falls in the range $(0, 1/2)$ precisely when $1+\sqrt{2} < p < 3$, to prove the equivalent of Lemma 4.2:

Lemma 5.3. *Let u be the solution to*

$$\Box_S u = F, \qquad u_{|\Sigma^-} = f, \qquad \partial_{\tilde v} u_{|\Sigma^-} = g\ . \tag{5.5}$$

Then we have for $1+\sqrt{2} < p < 3$

$$\|u\|_X \lesssim \|F\|_N + E[f,g]\ . \tag{5.6}$$

The proof is similar (but easier) to that of Lemma 4.2 , where one uses (5.2) instead of (3.7) to bound the commutator term $\chi[\Box_S, Z^\gamma]u$. With Lemma 5.3 instead of Lemma 4.2, it is easy to see that the proof of the Strauss conjecture, Theorem 5.1, is similar to that of Theorem 4.1.

5.1. **Proof of Proposition 5.2.** As in [50], we want to rewrite the equation near infinity as $(\partial_t^2 + P)w = F$ so that P is elliptic and self-adjoint Laplacian with respect to $L^2(\mathbb{R}^3)$.

Recall that, in the (t, r, ω) coordinates, we have

$$\Box_S = -\Big(1-\frac{2M}{r}\Big)^{-1}\partial_t^2 + r^{-2}\partial_r r^2\Big(1-\frac{2M}{r}\Big)\partial_r + r^{-2}\Delta_\omega = -\Big(1-\frac{2M}{r}\Big)^{-1}(\partial_t^2 + Q)$$

where $-Q = \Big(1-\frac{2M}{r}\Big)r^{-2}\partial_r r^2\Big(1-\frac{2M}{r}\Big)\partial_r + \Big(1-\frac{2M}{r}\Big)r^{-2}\Delta_\omega$ is a self-adjoint operator with respect to the metric $\Big(1-\frac{2M}{r}\Big)^{-1}dx$. We see that $\Box_S u = G$ is equivalent to

$$(\partial_t^2 + Q)u = -\Big(1-\frac{2M}{r}\Big)G\ .$$

The self-adjoint operator we are seeking is

$$P = \Big(1-\frac{2M}{r}\Big)^{-1/2} Q \Big(1-\frac{2M}{r}\Big)^{1/2}$$

In conclusion, the above calculation tells us that

$$\Box_S u = G$$

is equivalent to the equation

$$(\partial_t^2 + P)w = -\Big(1-\frac{2M}{r}\Big)^{1/2} G\ , \tag{5.7}$$

with $w = \left(1 - \frac{2M}{r}\right)^{-1/2} u$ and P as above,

$$P = -\left(1 - \frac{2M}{r}\right)^{1/2} r^{-2}\partial_r r^2 \left(1 - \frac{2M}{r}\right)\partial_r \left(1 - \frac{2M}{r}\right)^{1/2} - \left(1 - \frac{2M}{r}\right) r^{-2}\Delta_\omega.$$

Since u is supported in $r \geq KM$ (where we have $\tilde{v} = t$), we can essentially reduce the problem to the problems studied in [50] and [58]. Since w vanishes for $r < KM$, we can easily extend P to a self-adjoint operator P_1 on $\mathbb{R}^3$ (e.g.,

$$P_1 = -hr^{-2}\partial_r r^2 h^2 \partial_r h - h^2 r^{-2}\Delta_\omega \tag{5.8}$$

where $h(r) = (1 - \tilde{\psi})\left(1 - \frac{2M}{r}\right)^{1/2} + \tilde{\psi}$ and $\tilde{\psi}$ is a radial function vanishing for $r > KM$ with $\tilde{\psi} = 1$ for $r < KM/2$).

In particular, we note that $P_1^{1/2} \in S^1_{hom}$ has symbol

$$p_1 = h^2\sqrt{p} + \frac{1}{r}e$$

Here $\sqrt{p}$ is the symbol of $\sqrt{-\Delta}$ and $e \in S^0_{hom}$, $e \equiv 0$ for $r < KM/2$.

We will need the following lemma, which asserts that $P_1^{1/2}$ behaves like ∇_x in the appropriate function spaces:

Lemma 5.4. *The following estimates hold in $\mathbb{R}^3$:*

$$\|P_1^{1/2} v\|_{l_\infty^{-1/2}(L^2)} \lesssim \|\nabla_x v\|_{l_\infty^{-1/2}(L^2)} + \|v\|_{l_\infty^{-3/2}(L^2)} \tag{5.9}$$

$$\|P_1^{s/2} u\|_{L^2} \simeq \|u\|_{\dot{H}^s}, \qquad s \in [-1, 1]. \tag{5.10}$$

Proof. Let K_e denote the kernel of the operator associated to the error $\frac{1}{r}e$. For (5.9) it is enough to prove that

$$\| \int K_e(x,y)v(y)dy\|_{l_\infty^{-1/2}(L^2)} \lesssim \|v\|_{l_\infty^{-3/2}(L^2)}$$

Fix a dyadic region $A_k = \{2^{k-1} \leq |x| \leq 2^k\}$. When $|y| \approx 2^k$, we easily get the bound

$$\| \int_{2^{k-2} \leq |y| \leq 2^{k+1}} K_e(x,y)v(y)dy\|_{L^2(A_k)} \lesssim \|r^{-1}v\|_{L^2(2^{k-2} \leq r \leq 2^{k+1})} \lesssim 2^{k/2}\|v\|_{l_\infty^{-3/2}(L^2)} \tag{5.11}$$

When $|y - x| \gg 1$, we use the bound

$$|K_e(x,y)| \lesssim |x-y|^{-n}|x+y|^{-1}, \qquad |x \pm y| \geq 1$$

which comes from $e \in S^0_{hom}$ and the self-adjointess of the operator.

By Hölder's inequality we get for $j > k + 2$:

$$| \int_{2^{j-1} \leq |y| \leq 2^j} K_e(x,y)v(y)dy| \lesssim | \int_{2^{j-1} \leq |y| \leq 2^j} 2^{-4j} v(y)dy| \lesssim 2^{-j}\|v\|_{l_\infty^{-3/2}(L^2)}$$

Similarly for $j < k-1$ we obtain the bound

$$|\int_{2^{j-1}\le |y|\le 2^j} K_e(x,y)v(y)dy| \lesssim 2^{3j-4k}\|v\|_{l_\infty^{-3/2}(L^2)}$$

Thus after summation we obtain the pointwise estimate

$$|\int_{|y-x|\gg 1} K_e(x,y)v(y)dy| \lesssim 2^{-k}\|v\|_{l_\infty^{-3/2}(L^2)}$$

from which (5.9) follows immediately, taking also (5.11) into account.

By interpolation and duality, we need only to prove the estimate (5.10) for the special case where $s=1$.

For $h(r) = (1-\tilde\psi)\left(1-\frac{2M}{r}\right)^{1/2}+\tilde\psi$, by choosing K large enough , we have

$$1/2 \le h \le 2\ ,|h'| \le C/r\ ,|\partial_r h^{-1}| \le C/r\ . \tag{5.12}$$

By the expression of P_1 (5.8), it is easy to see that

$$\|P_1^{1/2}u\|_{L^2}^2 = \int_{\mathbb{R}^3} h^2\left(|\partial_r hu|^2 + |\not\nabla u|^2\right)dx \simeq \|\nabla(hu)\|_{L^2}^2\ .$$

However, the equivalence between $\|\nabla u\|_{L^2}$ and $\|\nabla(hu)\|_{L^2}$ can be seen by the Hardy inequality and (5.12).

□

The proof of Proposition 5.2 is reduced to the following proposition.

Proposition 5.5. *Let w solves the equation $(\partial_t^2+P_1)w = G = G_1+G_2$ with initial data (w_0,w_1) on $\{t=0\}$, we have*

$$\begin{aligned}\|r^{3/2-4/q-s}w\|_{L^q_{t,r\ge1}L^2_\omega} &\le C(\|w_0\|_{\dot H^s}+\|w_1\|_{\dot H^{s-1}}\\ &+\|r^{-1/2-s}G_1\|_{L^1_{t,r}L^2_\omega}+\|\langle r\rangle^{3/2-s+\delta}G_2\|_{L^2_{t,x}}),\end{aligned} \tag{5.13}$$

for any $q\ge 2$, $1/2-1/q<s<1/2$.

Let us give the proof of Proposition 5.2, based on Proposition 5.5.

Proof of Proposition 5.2. Since u vanishes in the region $\{r<KM\}$, $\Box_S u = F$ with initial data (u_0,u_1) is equivalent to

$$(\partial_t^2+P_1)w = G$$

with $w = \left(1-\frac{2M}{r}\right)^{-1/2}u$ with initial data $(w_0,w_1) = (\left(1-\frac{2M}{r}\right)^{-1/2}f, \left(1-\frac{2M}{r}\right)^{-1/2}g)$ and $G = -\left(1-\frac{2M}{r}\right)^{1/2}F$. Noting the support property of u, there is a cutoff function ϕ with support in $\{r<KM\}$ and $\phi=1$ for $r<(K-1)M$, such that $u=(1-\phi)u$ and so

$$w = \left(1-\frac{2M}{r}\right)^{-1/2}(1-\phi)u\ .$$

Then by the fractional Leibniz rule (see e.g. Lemma 2.7 of [58]), we have

$$\|w\|_{\dot H^s} \le C\left\|\left(1-\frac{2M}{r}\right)^{-1/2}(1-\phi)\right\|_{L^\infty\cap \dot W^{1,3}}\|u\|_{\dot H^s} \le C\|u\|_{\dot H^s}, s\in[-1,1]\ . \tag{5.14}$$

Thus, by Proposition 5.5, we get

$$\begin{aligned}\|r^{3/2-4/q-s}u\|_{L^q_{\tilde{v},r}L^2_\omega(\mathcal{M})} &\leq \|r^{3/2-4/q-s}w\|_{L^q_{t,r\geq 1}L^2_\omega} \\ &\leq C(\|w_0\|_{\dot{H}^s}+\|w_1\|_{\dot{H}^{s-1}} \\ &\quad +\|r^{-1/2-s}G_1\|_{L^1_{t,r}L^2_\omega}+\|\langle r\rangle^{3/2-s+\delta}G_2\|_{L^2_{t,x}}) \\ &\leq C(\|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-1}} \\ &\quad +\|r^{-1/2-s}F_1\|_{L^1_{t,r}L^2_\omega}+\|\langle r\rangle^{3/2-s+\delta}F_2\|_{L^2_{t,x}}),\end{aligned}$$

where we have used the inequality (5.14) for w_0 and w_1. This completes the proof. ∎

Now we turn to the proof of the Proposition 5.5. Let w_{hom} be the solution to the homogeneous problem with initial data (w_0, w_1). Inequality (2.3) implies in particular that

$$\|\nabla w_{hom}\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty(L^2_t L^2_x)} + \|w_{hom}\|_{l^{-3/2}_\infty(L^2_t L^2_x)} \lesssim \|w_0\|_{\dot{H}^1} + \|w_1\|_{L^2}. \tag{5.15}$$

Since ∂_t and P_1 commute with $P_1^{-1/2}$, we obtain

$$\|\nabla P_1^{-1/2} w_{hom}\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty(L^2_t L^2_x)} + \|P_1^{-1/2} w_{hom}\|_{l^{-3/2}_\infty(L^2_t L^2_x)} \lesssim \|P_1^{-1/2} w_0\|_{\dot{H}^1} + \|P_1^{-1/2} w_1\|_{L^2}$$

which after using Lemma 5.4 yields

$$\|w_{hom}\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty(L^2_t L^2_x)} \lesssim \|w_0\|_{L^2} + \|w_1\|_{\dot{H}^{-1}}. \tag{5.16}$$

The inhomogeneous part $w - w_{hom}$ has vanishing initial data, so we can use the estimates (3.8) and (3.9) to bound it. We thus obtain, using (5.15) and (5.16):

$$\|\nabla w\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty(L^2_t L^2_x)} \lesssim \|w_0\|_{\dot{H}^1} + \|w_1\|_{L^2} + \|G\|_{L^1_t L^2_x + l^{1/2}_1(L^2_t L^2_x)}$$

$$\|w\|_{L^\infty_t L^2_x \cap l^{-1/2}_\infty(L^2_t L^2_x)} \lesssim \|w_0\|_{L^2} + \|w_1\|_{\dot{H}^{-1}} + \|G\|_{l^{1/2}_1(L^2_t \dot{H}^{-1}_x) + L^1_t \dot{H}^{-1}_x}$$

The proof of Proposition 5.5 now follows as in Proposition 3.2 through interpolation and the use of the trace lemma.

References

[1] S. Alinhac, *On the Morawetz-Keel-Smith-Sogge inequality for the wave equation on a curved background*, Publ. Res. Inst. Math. Sci. **42** (2006), 705–720.

[2] S. Alinhac, *Energy multipliers for perturbations of the Schwarzschild metric*, Comm. Math. Phys. **288** (2009). 199–224.

[3] L. Andersson, P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, arXiv:0908.2265

[4] A. Bachelot, J.-P. Nicolas, *Équation non linéaire de KleinGordon dans des métriques de type Schwarzschild*. (French) [Nonlinear Klein-Gordon equation in Schwarzschild-type metrics] C. R. Acad. Sci. Paris Sr. I Math. **316** (1993), no. 10, 1047–1050. MR1222970

[5] J. Bergh, J. Löfström: *Interpolation spaces. An introduction* Grundlehren der Mathematischen Wissenschaften, No. **223**. Springer-Verlag, Berlin-New York, 1976. x+207 pp.

[6] P. Blue, A. Soffer, *Semilinear wave equations on the Schwarzschild manifold I: local decay estimates*. Advances in Differential Equations **8** (2003), 595–614.

[7] P. Blue, A. Soffer, *The wave equation on the Schwarzschild metric II: Local decay for the spin-2 Regge-Wheeler equation*, J. Math. Phys., **46** (2005), 9pp.

[8] P. Blue, J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space*, Comm. Math. Phys. **268** (2006), no. 2, 481–504. MR2259204

[9] J.-F. Bony, D. Häfner, *The semilinear wave equation on asymptotically Euclidean manifolds*, Comm. Partial Differential Equations **35**(2010), no. 1, 23-67. MR2748617

[10] N. Burq, *Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: Global Strichartz estimates for nontrapping perturbations of the Laplacian [Comm. Partial Differential Equation* **25** *(2000), no. 11–12 2171–2183; MR1789924]*, Comm. Partial Differential Equations **28** (2003), 1675–1683.

[11] N. Burq, F. Planchon, J. G. Stalker, A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. **53** (2004), 1665–1680.

[12] D. Catania, V. Georgiev, *Blow-up for the semilinear wave equation in the Schwarzschild metric.* Differential Integral Equations **19** (2006), no. 7, 799–830.

[13] S. Chandrasekhar. *The Mathematical Theory of Black Holes.* Clarendon Press, Oxford, MA, 1983.

[14] M. Dafermos, I. Rodnianski, *Small-amplitude nonlinear waves on a black hole background*, J. Math. Pures Appl. **84** (2005), no. 9, 1147–1172. MR2162222

[15] M. Dafermos, I. Rodnianski, *A note on energy currents and decay for the wave equation on a Schwarzschild background.* arXiv:0710.0171

[16] M. Dafermos, I. Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes.* Comm. Pure Appl. Math. **62** (2009), no. 7, 859–919. MR2527808

[17] M. Dafermos, I. Rodnianski, *Lectures on black holes and linear waves*, arXiv:0811.0354

[18] M. Dafermos, I. Rodnianski, *Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry*, arXiv:1010.5132

[19] M. Dafermos, I. Rodnianski, *The black hole stability problem for linear scalar perturbations*, arXiv:1010.5137

[20] Y. Du, J. Metcalfe, C. D. Sogge, Y. Zhou, *Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in 4-dimensions*, Comm. Partial Differential Equations **33** (2008), no. 7-9, 1487–1506. MR2450167

[21] D. Fang, C. Wang, *Weighted Strichartz Estimates with Angular Regularity and their Applications*, Forum Math. **23** (2011), no. 1, 181–205. MR2769870

[22] V. Georgiev, H. Lindblad, C. D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119** (1997), no. 6, 1291–1319. MR1481816

[23] R. T. Glassey, *Existence in the large for $\Box u = F(u)$ in two space dimensions*, Math. Z. **178** (1981), no. 2, 233–261. MR0631631

[24] R. T. Glassey, *Finite-time blow-up for solutions of nonlinear wave equations* , Math. Z. **177** (1981), no. 3, 323–340.

[25] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.

[26] K. Hidano, *Morawetz-Strichartz estimates for spherically symmetric solutions to wave equations and applications to semilinear Cauchy problems*, Differential Integral Equations **20** (2007), no. 7, 735–754. MR2333654

[27] K. Hidano, J. Metcalfe, H. F. Smith, C. D. Sogge, Y. Zhou, *On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2789–2809. MR2584618

[28] K. Hidano, K. Yokoyama, *A remark on the almost global existence theorems of Keel, Smith, and Sogge*, Funkcial. Ekvac. **48** (2005), 1–34.

[29] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions* , Manuscripta Math. **28** (1979), no. 1-3, 235–268. MR0535704

[30] M. Keel, H. Smith, C. D. Sogge, *Almost global existence for some semilinear wave equations*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 265–279.

[31] M. Keel, H. F. Smith, C. D. Sogge, *Almost global existence for quasilinear wave equations in three space dimensions*, J. Amer. Math. Soc. **17** (2004), no. 1, 109–153 (electronic).

[32] C. E. Kenig, G. Ponce, L. Vega, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal. **127** (1995), 204–234.

[33] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation.* Commun. Pure Appl. Math. **38** (1985), 321–332.

[34] H. Lindblad, C. D. Sogge, *Long-time existence for small amplitude semilinear wave equations*, Amer. J. Math. **118** (1996), no. 5, 1047–1135. MR1408499

[35] J. Marzuola, J. Metcalfe, D. Tataru, M. Tohaneanu, *Strichartz estimates on Schwarzschild black hole backgrounds*, Comm. Math. Phys. **293** (2010), no. 1, 37–83. MR2563798

[36] J. Metcalfe, C. D. Sogge, *Long time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal. **38** (2006), no. 1, 188–209. MR2217314

[37] J. Metcalfe, C. D. Sogge, *Global existence of null-form wave equations in exterior domains*, Math. Z. **256** (2007), 521–549.

[38] J. Metcalfe, D. Tataru, *Global parametrices and dispersive estimates for variable coefficient wave equations*, Math. Ann. **353** (2012), no. 4, 1183-1237. MR2944027

[39] J. Metcalfe, D. Tataru, M. Tohaneanu, *Price's law on nonstationary space-times*, Adv. Math. **230** (2012), no. 3, 995–1028. MR2921169

[40] C. S. Morawetz, *Exponential decay of solutions of the wave equation*, Comm. Pure Appl. Math. **19** (1966), 439–444.

[41] C. S. Morawetz, *Time decay for the nonlinear Klein-Gordon equations*, Proc. R. Soc. Lond. Ser. A **306** (1968), 291–296.

[42] C. S. Morawetz, *Decay for solutions of the exterior problem for the wave equation*, Comm. Pure Appl. Math. **28** (1975), 229–264.

[43] J.-P. Nicolas, *Nonlinear Klein-Gordon equation on Schwarzschild-like metrics*, J. Math. Pures Appl. **74** (1995), no. 1, 35–58.

[44] J. V. Ralston, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math. **22** (1969), 807–923.

[45] J. Schaeffer, *The equation* $\Box u = |u|^p$ *for the critical value of* p, Proc. Roy. Soc. Edinburgh Sect. A **101** (1985), no. 1-2, 31–44. MR0824205

[46] T. C. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations, **52** (1984), 378–406. MR0744303

[47] H. F. Smith, C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations **25** (2000), 2171–2183.

[48] H. F. Smith, C. D. Sogge, C. Wang, *Strichartz estimates for Dirichlet-wave equations in two dimensions with applications*, Trans. Amer. Math. Soc. **364** (2012), 3329-3347. MR2888248.

[49] C. D. Sogge, *Lectures on non-linear wave equations. Second edition.* International Press, Boston, MA, 2008. MR2455195

[50] C. D. Sogge, C. Wang, *Concerning the wave equation on asymptotically Euclidean manifolds*, J. Anal. Math. **112** (2010), no. 1, 1–32. MR2762995

[51] J. Sterbenz, *Angular regularity and Strichartz estimates for the wave equation*, Int. Math. Res. Not. (2005), 187–231. With an appendix by I. Rodnianski.

[52] W. A. Strauss, *Dispersal of waves vanishing on the boundary of an exterior domain*, Comm. Pure Appl. Math. **28** (1975), 265–278.

[53] W. A. Strauss, *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), 110–133.

[54] D. Tataru, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 795–807. MR1804518

[55] D. Tataru, M. Tohaneanu, *A local energy estimate on Kerr black hole backgrounds*. Int. Math. Res. Not. IMRN 2011, no. 2, 248–292. MR2764864

[56] H. Triebel: *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, **18**. North-Holland Publishing Co., Amsterdam-New York, 1978. 528 pp.

[57] M. Tohaneanu, *Strichartz estimates on Kerr black hole backgrounds*. Trans. Amer. Math. Soc. **364** (2012), no. 2, 689–702. MR2846348

[58] C. Wang, X. Yu, *Concerning the Strauss conjecture on asymptotically Euclidean manifolds*, J. Math. Anal. Appl. **379** (2011), no. 2, 549–566. MR2784342

[59] C. Wang, X. Yu, *Recent works on the Strauss conjecture*. Recent Advances in Harmonic Analysis and Partial Differential Equations, 235–256, Contemp. Math., 581, Amer. Math. Soc., Providence, RI, 2012

[60] B. Yordanov, Q. S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. **231** (2006), no. 2, 361–374.

[61] X. Yu, *Generalized Strichartz estimates on perturbed wave equation and applications on Strauss conjecture*, Differential Integral Equations **24** (2011), no. 5-6, 443–468. MR2809616

[62] Y. Zhou, *Cauchy problem for semilinear wave equations with small data in four space dimensions*, J. Partial Differential Equations **8** (1995), no. 2, 135–144. MR1331521

[63] Y. Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chinese Ann. Math. Ser. B, **28** (2007), no. 2, 205–212.

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