POINTWISE DECAY FOR THE MAXWELL FIELD ON BLACK HOLE SPACE-TIMES

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Abstract. In this article we study the pointwise decay properties of solutions to the Maxwell system on a class of nonstationary asymptotically flat backgrounds in three space dimensions. Under the assumption that uniform energy bounds and a weak form of local energy decay hold forward in time, we establish peeling estimates, as well as a t^{-4} rate of decay on compact regions for all the components of the Maxwell tensor.

1. Introduction

In this article we consider the question of pointwise decay for solutions to the Maxwell system with localized initial data. The class of backgrounds we are interested in are certain asymptotically flat black hole backgrounds, e.g of Schwarzschild/Kerr type and perturbations thereof. However, the type of results we obtain in this article treat a compact set essentially as a black box, so they also apply in other settings. Our interest in this problem originates from general relativity, where the Maxwell (or spin 1) system is a linearized model of the Einstein Equations that captures some of the difficulties not present in the scalar wave equation (or spin 0) case.

The main idea of this article is that the pointwise decay bounds are a consequence of local energy decay estimates for the same Maxwell system, even though the local energy decay bounds are invariant with respect to time translations, while the pointwise decay bounds are not. This fits into the philosophy that the local energy decay estimates are the core decay estimates, and the other types of decay estimates (e.g. Strichartz, pointwise) are derived bounds. In the context of the Schrödinger equation on asymptotically flat space-times, this approach was developed in [29], [20]. More recently, the same philosophy was implemented in the context of the scalar wave equation, beginning with [22]. The case of the scalar wave equation on black hole space times is discussed in what follows.

We begin with local energy estimates for solutions to the scalar wave equation $\Box_g u = f$ on Schwarzschild and Kerr manifolds, which have been recently established by various authors ([4], [5], [6], [8], [9], [21] for Schwarzschild, [30], [10], [1] for Kerr with small angular momentum, and [11], [12], [13] for Kerr with $|a| < M$). The transition from local energy decay to Strichartz estimates was considered in [21], [31]. The key result that sharp decay bounds (Price's Law [25]) follow from the local energy decay was first obtained in [29] for stationary space-times, using time Fourier transform and resolvent

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analysis, and then in the nonstationary case in [23], by using more robust methods based on the classical vector field method. (See also [14], [15] for a more refined Fourier based analysis applied to Schwarzschild space-times.)

The main result in the present article is the exact counterpart of [23] in the context of the Maxwell system, and asserts that local energy decay implies sharp[1] pointwise decay bounds. These can be seen as Price's law in the Maxwell setting; indeed, [26] conjectures a decay rate of t^{-5} in compact regions for the Maxwell system on the Schwarzschild metric.

Since our result is a conditional one, it is useful to review where we stand as far as local energy decay estimates are concerned. With regards to the Maxwell system on Schwarzschild, a class of local energy estimates (as well as some partial pointwise rates of decay) were established in [3] for solutions to the homogeneous system with no charge. For solutions to the homogeneous system on Kerr spacetimes with small angular momentum $|a| \ll M$ there is recent work [2] that establishes some local energy estimates and uniform energy bounds.

For the inhomogeneous system with charges, the article [28] provides local energy estimates in a variety of spherically symmetric spacetimes, including Schwarzschild. This is the context where the results in the present paper directly apply. We expect the analogous estimates for the Kerr spacetimes and small perturbations thereof to also hold, in which case the same decay results would be true. We remark that the results in [2] cannot be used directly in our present context, as they only deal with solutions to the homogeneous equation; however, if one had the appropriate inhomogeneous version of the bounds in [2], that would suffice.

While there are substantial similarities between our present result for the Maxwell system and our earlier work [23] for the scalar wave equation, there are also some significant differences. Some of these differences are of a technical nature and stem from the fact that we are dealing with a first order hyperbolic system rather than with a first order scalar wave equation.

However, there is also a significant conceptual difference, which is that even in the simplest stationary problem one has nontrivial zero modes to deal with. These zero mode components are parametrized by the electric, respectively the magnetic charge, of the system, which are conserved quantities. In spherical symmetry the problem simplifies considerably in that these modes correspond exactly to the radial part of the Maxwell tensor, and thus can be easily eliminated. Instead of taking this easy way out, here we develop an approach that relies neither on the radiality nor on the stationarity of the metric.

A natural follow-up question to ask here would be whether our approach here generalizes to the spin 2 case, i.e. to linearized gravity. One obvious additional difficulty there is that there are more zero modes present, and it is not clear to us whether these modes can be tracked in the same way as in the present paper, given a general (i.e. non Kerr) black hole space-time.

[1] At least for $r \geq \frac{t}{2}$; understanding what happens in the interior of a small cone seems to be a more delicate matter.

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2. Notation and setup

2.1. **Notations.** We use $(t = x_0, x)$ for the coordinates in $\mathbb{R}^{1+3}$. We use Latin indices $i, j = 1, 2, 3$ for spatial summation and Greek indices $\alpha, \beta = 0, 1, 2, 3$ for space-time summation. In $\mathbb{R}^3$ we also use polar coordinates $x = r\omega$ with $\omega \in \mathbb{S}^2$. By $\langle r \rangle$ we denote a smooth radial function which agrees with r for large r and satisfies $\langle r \rangle \geq 2$. We consider a partition of $\mathbb{R}^3$ into the dyadic sets $A_R = \{\langle r \rangle \approx R\}$ for $R \geq 1$, with the obvious change for $R = 1$.

2.2. **Space-times.** We are interested in uniformly smooth asymptotically flat Lorentzian space-times (M, g) in either $M = \mathbb{R}^+ \times \mathbb{R}^3$ or an exterior region of the form $M = \mathbb{R}^+ \times \mathbb{R}^3 \setminus B(0, R_0)$. To set a proper orientation for our space-time, we make the following assumption:

(i) The level sets $t = const$ are space-like.

To describe the regularity of the coefficients of the metric, we use the following sets of vector fields:

$$\partial = \{\partial_t, \partial_i\}, \qquad \Omega = \{x_i\partial_j - x_j\partial_i\}, \qquad S = t\partial_t + x\partial_x,$$

namely the generators of translations, rotations and scaling. We set $Z = \{T, \Omega, S\}$. Then we define the classes $S^Z(r^k)$ of functions in $\mathbb{R}^+ \times \mathbb{R}^3$ by

$$a \in S^Z(r^k) \iff |Z^j a(t, x)| \leq c_j \langle r \rangle^k, \quad j \geq 0.$$

By $S^Z_{rad}(r^k)$ we denote spherically symmetric functions in $S^Z(r^k)$.

This leads us to our second main assumption.

(ii) (M, g) is asymptotically flat.

Here, for the purpose of the present paper, we make the following definition:

Definition 2.1. *We say that g is asymptotically flat if it has the form*

$$g = m + g_{sr} + g_{lr},$$

where m stands for the Minkowski metric, g_{lr} is a stationary long range spherically symmetric component, with $S^Z_{rad}(r^{-1})$ coefficients, of the form

$$g_{lr} = g_{lr,tt}(r)dt^2 + g_{lr,tr}(r)dtdr + g_{lr,rr}(r)dr^2 + g_{lr,\omega\omega}(r)r^2d\omega^2$$

and g_{sr} is a short range component of the form

$$g_{sr} = g_{sr,tt}dt^2 + 2g_{sr,ti}dtdx_i + g_{sr,ij}dx_idx_j$$

with $S^Z(r^{-2})$ coefficients which also satisfy

$$\partial g_{sr} \in S^Z(r^{-3}). \tag{2.1}$$

This definition is set to match the setup of relativistic space-times, e.g. Schwarzschild and Kerr. In that context, the $O(r^{-1})$ radial part of the metric is associated to mass, while the $O(r^{-2})$ nonradial terms are associated to the angular momentum. Having accurate decay rates for the metric perturbation at infinity is essential in this work; indeed, these decay rates, rather that the local behavior of the metric, are the factor which determines the exact decay rates for both scalar and electromagnetic waves. We also remark that, in contrast with our previous result for the wave equation, we require that the derivative of the short range component gains one extra order of decay. This will prove to be crucial in gaining extra decay for the radial part in Section 6, as well as obtaining the correct bounds for the curvature tensor in Section 7.

Our decay results are expressed relative to the distance to the Minkowski null cone $\{t = |x|\}$. This can only be done provided that there is a null cone associated to the metric g which is within $O(1)$ of the Minkowski null cone. However, in general the long range component of the metric produces a logarithmic correction to the cone. This issue can be remedied via a change of coordinates that roughly corresponds to using Regge-Wheeler coordinates in Schwarzschild/Kerr near spatial infinity; see [29]. This is related to the fact that our asymptotic flatness condition is stable with respect to a class of changes of coordinates χ of the form

$$\chi = \chi_{lr} + \chi_{sr}$$

where χ_{lr} is radial and satisfies $\nabla\chi_{lr} - I \in S^Z(r^{-1})$ while $\nabla\chi_{sr} \in S^Z(r^{-2})$. This class allows for logarithmic cone corrections. Indeed, after a further conformal transformation, the metric g is reduced to a normal form where

$$g_{lr} = g_\omega(r) r^2 d\omega^2, \qquad g_\omega \in S^Z_{rad}(r^{-1}). \tag{2.2}$$

In this context, we can replace $\Box_g$ by an operator of the form

$$P = \Box + Q \tag{2.3}$$

where $\Box$ denotes the d'Alembertian in the Minkowski metric and the perturbation Q has the form

$$Q = g^\omega \Delta_\omega + \partial_\alpha g^{\alpha\beta}_{sr} \partial_\beta + V, \quad g^{\alpha\beta}_{sr} \in S^Z(r^{-2}),\ g^\omega \in S^Z_{rad}(r^{-3}),\ V \in S^Z(r^{-3}). \tag{2.4}$$

See [29] and [23] for more details.

We call these coordinates normalized coordinates. Most of the analysis in the paper is done in normalized coordinates and with g in normalized form.

Finally, concerning the local properties of the metric we make either one of the following assumptions:

$(iii)_a$ *(regular space-time)* $M = \mathbb{R}^+ \times \mathbb{R}^3$.

$(iii)_b$ *(black hole space-time)* $M = \mathbb{R}^+ \times \mathbb{R}^3 \setminus B(0, R_0)$ and the lateral boundary $\mathbb{R} \times \partial B(0, R_0)$ is outgoing space-like.

One could consider also other settings, e.g. exterior space-times $M = \mathbb{R}^+ \times \mathbb{R}^3 \setminus \partial B(0, R_0)$ with various boundary conditions on the time-like boundary $\mathbb{R} \times \partial B(0, R_0)$.

2.3. **The Maxwell system.** In spacetimes as above, we consider a Maxwell field F, which is an antisymmetric $(0,2)$-tensor field on a Lorentzian manifold (M,g) satisfying the Maxwell equations:

$$dF = G_1, \qquad d*F = G_2. \tag{2.5}$$

In the physical context one disallows magnetic currents and sets $G_1 = 0$. However, mathematically it is more convenient to work in a symmetric setting and allow both G_1 and G_2 to be nonzero.

We will assume that the initial data $F(0)$ at time $t = 0$ is smooth and compactly supported. The inhomogeneous terms G_1 and G_2 should satisfy the compatibility conditions

$$dG_1 = dG_2 = 0,$$

as well as be supported in the forward cone $C = \{t \geq r - R_1\}$ for some $R_1 > 0$.

For comparison purposes, we also state the corresponding result for the scalar wave equation,

$$\Box_g u = f \tag{2.6}$$

with initial data $u[0] = (u(0), \partial_t u(0))$ at time $t = 0$. This is the problem considered in our preceding paper [23], to which we will refer repeatedly here.

2.4. **Local energy norms.** We now introduce our local energy norms. For a scalar function u we define

$$\begin{aligned}
\|u\|_{LE} &= \sup_R \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}^+\times A_R)},\\
\|u\|_{LE[t_0,t_1]} &= \sup_R \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2([t_0,t_1]\times A_R)},\\
\|u(t_0,\cdot)\|_{\mathcal{LE}} &= \sup_R \|\langle r\rangle^{-\frac{1}{2}} u(t_0,\cdot)\|_{L^2(A_R)},
\end{aligned} \tag{2.7}$$

where the last norm applies at fixed time. Their H^1 counterparts were also used in [23] in the study of the scalar wave equation (2.6):

$$\begin{aligned}
\|u\|_{LE^1} &= \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE},\\
\|u\|_{LE^1[t_0,t_1]} &= \|\nabla u\|_{LE[t_0,t_1]} + \|\langle r\rangle^{-1} u\|_{LE[t_0,t_1]},\\
\|u(t_0)\|_{\mathcal{LE}^1} &= \|\nabla_x u(t_0,\,\cdot)\|_{\mathcal{LE}} + \|\langle r\rangle^{-1} u(t_0,\,\cdot)\|_{\mathcal{LE}}.
\end{aligned} \tag{2.8}$$

Here and in the rest of the paper we use the abbreviation $\nabla = \nabla_{t,x}$.

The corresponding dual type spaces, used for the source terms, are:

$$\begin{aligned}
\|f\|_{LE^*} &= \sum_R \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}_+\times A_R)},\\
\|f\|_{LE^*[t_0,t_1]} &= \sum_R \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2([t_0,t_1]\times A_R)},\\
\|f(t_0,\cdot)\|_{\mathcal{LE}^*} &= \sum_R \|\langle r\rangle^{\frac{1}{2}} f(t_0,\cdot)\|_{L^2(A_R)}.
\end{aligned} \tag{2.9}$$

We also define similar norms for higher Sobolev regularity

$$\|u\|_{LE^{1,k}} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{LE^1},$$

$$\|u\|_{LE^{1,k}[t_0,t_1]} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{LE^1[t_0,t_1]},$$

$$\|u\|_{LE^k} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{LE},$$

$$\|u\|_{LE^k[t_0,t_1]} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{LE[t_0,t_1]},$$

respectively

$$\|f\|_{LE^{*,k}} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{LE^*},$$

$$\|f\|_{LE^{*,k}[t_0,t_1]} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{LE^*[t_0,t_1]}.$$

For a triplet $\Lambda = (i, j, k)$ of multi-indices i, j and k we denote $|\Lambda| = |i| + 3|j| + 9k$ and

$$u^\Lambda = \partial^i \Omega^j S^k u, \qquad u^{\leq m} = (u^\Lambda)_{|\Lambda|\leq m}.$$

The choice of weights here follows [23], but is somewhat arbitrary. The goal is to enable us to treat all differentiated terms with better decaying coefficients perturbatively in the Ωu equation, and to also treat all Ω terms with better decaying coefficients perturbatively in the Su equation.

We also define, for any norm Y,

$$\|u^{\leq m}\|_Y = \sum_{|\Lambda|\leq m} \|u^\Lambda\|_Y.$$

In the case of black hole space times one also needs to contend with trapping. Fortunately, for our purposes here one does not need to pay too much attention to that, and it suffices to use a rough regularity analysis.

Definition 2.2. *a) We say that the scalar wave evolution* (2.6) *has the local energy decay property if the following estimate holds:*

$$\|u\|_{LE^{1,k}[t_0,\infty)} \leq c_k(\|\nabla u(t_0)\|_{H^k} + \|f\|_{LE^{*,k}[t_0,\infty)}), \qquad k \geq 0. \tag{2.10}$$

b) We say that the scalar wave evolution (2.6) *has the weak local energy decay property if the following estimate holds:*

$$\|u\|_{LE^{1,k}[t_0,\infty)} \leq c_k(\|\nabla u(t_0)\|_{H^{k+1}} + \|f\|_{LE^{*,k+1}[t_0,\infty)}), \qquad k \geq 0 \tag{2.11}$$

in either $\mathbb{R} \times \mathbb{R}^3$ or in the exterior domain (black hole) case.

The first definition applies for the nontrapping case. The second one is for the black hole case, where we allow for a loss of one derivative to account for trapping effects. We remark that in the presence of hyperbolic trapping this loss is much more than is required. Indeed, generally hyperbolic trapping merely produces a logarithmic loss, and that only near the trapped set. But that is not so relevant to our purposes here, so we content ourselves with the more relaxed bound (2.11).

We also give the following definition (see [23] for the motivation):

Definition 2.3. *We say that the problem $\Box_g u = f$ satisfies stationary local energy decay bounds if on any time interval $[t_0, t_1]$ and $k \geq 0$ we have*

$$\|u\|_{LE^{1,k}[t_0,t_1]} \lesssim_k \|\nabla u(t_0)\|_{H^k} + \|\nabla u(t_1)\|_{H^k} + \|f\|_{LE^{*,k}[t_0,t_1]} + \|\partial_t u\|_{LE^{0,k}[t_0,t_1]}. \tag{2.12}$$

Let us also mention that all the definitions above can be easily extended to a vector $\vec{u}$ of functions by considering each component separately.

For the Maxwell tensor F, we need to slightly modify our energy norms. Using Cartesian coordinates, define

$$\begin{aligned} \|F\|_{LE} &= \sum_{\alpha,\beta} \|F_{\alpha\beta}\|_{LE}, \\ \|F(t_0)\|_{\mathcal{LE}} &= \sum_{\alpha,\beta} \|F(t_0,\cdot)_{\alpha\beta}\|_{\mathcal{LE}}, \end{aligned} \tag{2.13}$$

and the dual norms

$$\begin{aligned} \|G\|_{LE^*} &= \sum_{\alpha,\beta,\gamma} \|G_{\alpha\beta\gamma}\|_{LE^*}, \\ \|G(t_0)\|_{\mathcal{LE}^*} &= \sum_{\alpha,\beta,\gamma} \|G(t_0,\cdot)_{\alpha\beta\gamma}\|_{\mathcal{LE}^*}. \end{aligned} \tag{2.14}$$

We will also need to define higher energy norms of the tensor F. Geometrically it makes the most sense to commute the system (2.5) with Lie derivatives of vector fields, which we will denote by $\mathcal{L}_X$. Given a set of vector fields A, a norm Y, a tensor W and a positive integer l, define

$$\|\mathcal{L}_A W\|_Y = \sum_{X\in A} \|\mathcal{L}_X W\|_Y$$
$$\mathcal{L}_{A^l} W = \{\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_l} W : X_1 \cdots X_l \in A\}.$$

Keeping the analogy with the scalar case, we also define the higher norms associated to translations

$$\|F\|_{LE^k} = \sum_{l\leq k} \|\mathcal{L}_{\partial^l} F\|_{LE}$$

and similarly for $\mathcal{LE}^k$ and their duals. We remark that the role of the LE^1 norms in the scalar case is now played by the LE norms for the Maxwell system.

On $\{t = t_0\}$ slices, we define the higher regularity norm for $k \geq 0$

$$E^k(t_0) = \sum_{l\leq k} \|\mathcal{L}_{\partial^l} F(t_0)\|_{L^2}, \qquad E(t_0) = E^0(t_0) \tag{2.15}$$

We will now distinguish between the radial and nonradial parts of the tensor, as they will have different rates of decay. This is where things are different from the scalar case, and this is caused by the zero modes associated to the electric and magnetic charges. Precisely, with an LE^* type source, one can drive up the charge inside the cone, and thus eliminate any chance for local energy decay. One remedy for this would be to factor out the charges. This works well for spherically symmetric space-times, where the charges correspond exactly to radial modes, but in general this strategy seems to be unfeasible.

However, the radial mode does seem to carry the bulk of the charge near infinity. This motivates our present strategy, where we weigh the radial mode differently, in a way which is consistent with the estimates we already know from [28] to hold in spherically symmetric space-times. Precisely, our stronger weight for the radial components of the Maxwell tensor are strong enough in order to guarantee that the charge remains zero at infinity. It also allows us to obtain the radial components by integrating the source term from infinity.

For a function ψ, we will denote by $\overline{\psi}$ its zero spherical harmonic. Motivated by the fact that in the case of the Schwarzschild metric the zero modes are $d\omega^2$ and $r^{-2}dt\wedge dr$, we define

$$\overline{F} = \overline{F_{tr}}dt\wedge dr + \overline{F_{\phi\theta}}d\omega^2,$$

respectively

$$\overline{G} = \overline{G_{t\phi\theta}}dt\wedge d\omega^2 + \overline{G_{r\phi\theta}}dr\wedge d\omega^2.$$

We can now define the norms that we are mostly interested in

$$\|F\|_{LE_{\text{Max}}} = \|F\|_{LE} + \|\langle r\rangle\overline{F}\|_{LE},$$
$$\|F(t_0)\|_{\mathcal{LE}_{\text{Max}}} = \|F(t_0,\cdot)\|_{\mathcal{LE}} + \|\langle r\rangle\overline{F}(t_0,\cdot)\|_{\mathcal{LE}},$$

and for the inhomogeneous part

$$\|G\|_{LE^*_{\text{Max}}} = \|G\|_{LE^*} + \|\langle r\rangle\overline{G}\|_{LE^*},$$
$$\|G(t_0)\|_{\mathcal{LE}^*_{\text{Max}}} = \|G(t_0,\cdot)\|_{\mathcal{LE}^*} + \|\langle r\rangle\overline{G}(t_0,\cdot)\|_{\mathcal{LE}^*}.$$

Moreover, for a given $k\geq 0$, the higher regularity norms associated with Sobolev regularity are set to be:

$$\|F\|_{LE^k_{\text{Max}}} = \|F\|_{LE^k} + \|\langle r\rangle\bar{F}\|_{LE^k}$$
$$\|F(t_0)\|_{\mathcal{LE}^k_{\text{Max}}} = \|F(t_0)\|_{\mathcal{LE}^k} + \|\langle r\rangle\bar{F}(t_0)\|_{\mathcal{LE}^k}$$

respectively

$$\|G\|_{LE^{*,k}_{\text{Max}}} = \|G\|_{LE^{*,k}} + \|\langle r\rangle\bar{G}\|_{LE^{*,k}}$$
$$\|G\|_{\mathcal{LE}^{*,k}_{\text{Max}}} = \|G(t_0)\|_{\mathcal{LE}^{*,k}} + \|\langle r\rangle\bar{G}(t_0)\|_{\mathcal{LE}^{*,k}}.$$

Finally, for Λ a triplet as above, let

$$F^\Lambda = \mathcal{L}_{\partial^i}\mathcal{L}_{\Omega^j}\mathcal{L}_{S^k}F,$$

and

$$F^{\leq m} = (F^\Lambda)_{|\Lambda|\leq m}, \qquad \|F^{\leq m}\|_Y = \sum_{|\Lambda|\leq m}\|F^\Lambda\|_Y.$$

We now define the norms

$$\begin{aligned}
\|F^\Lambda\|_{LE_{\mathrm{Max}}} &= \|F^\Lambda\|_{LE} + \|\langle r\rangle \bar{F}^\Lambda\|_{LE},\\
\|F^\Lambda(t_0)\|_{\mathcal{LE}_{\mathrm{Max}}} &= \|F^\Lambda(t_0,\,\cdot)\|_{\mathcal{LE}} + \|\langle r\rangle \bar{F}^\Lambda(t_0,\,\cdot)\|_{\mathcal{LE}},\\
\|G^\Lambda\|_{LE^*_{\mathrm{Max}}} &= \|G^\Lambda\|_{LE^*} + \|\langle r\rangle \bar{G}^\Lambda\|_{LE^*},\\
\|G^\Lambda(t_0)\|_{\mathcal{LE}^*_{\mathrm{Max}}} &= \|G^\Lambda(t_0,\,\cdot)\|_{\mathcal{LE}^*} + \|\langle r\rangle \bar{G}^\Lambda(t_0,\,\cdot)\|_{\mathcal{LE}^*}.
\end{aligned}$$

We will assume that the following bounds hold:

Definition 2.4. *a) We say that the problem (2.5) has the local energy decay property if the following estimate holds for each $k \geq 0$:*

$$\sup_{t>t_0} E^k(t) + \|F\|_{LE^k_{Max}} \lesssim_k E^k(t_0) + \sum_{i=1}^{2} \|G_i\|_{LE^{*,k}_{Max}}. \tag{2.16}$$

b) We say that the problem (2.5) has the weak local energy decay property if the following estimate holds for each $k \geq 0$:

$$\sup_{t>t_0} E^k(t) + \|F\|_{LE^k_{Max}} \lesssim_k E^{k+1}(t_0) + \sum_{i=1}^{2} \|G_i\|_{LE^{*,k+1}_{Max}}. \tag{2.17}$$

Similarly to the case of the scalar wave equation, the first definition is adapted to the nontrapping case, while in the second we allow for a loss of a derivative to account for possible trapped geodesics.

We further comment on the choice of weights for the radial components, by first noting that for radial metrics the radial components uncouple and satisfy an equation which is essentially of the form $d(r^2\bar{F}) = r^2\bar{G}$, and the charge at infinity is the limit of $r^2\bar{F}$. Our weights require $\bar{F}$ to decay at least like r^{-2} at infinity, and $\bar{G}$ to decay at least like r^{-3} at infinity, with added integrability. Given the form of the radial equation, this exactly suffices in order to guarantee that the charge remains zero at infinity, and that $r^2\bar{F}$ can be obtained by integrating $r^2\bar{G}$ from infinity.

We also need an estimate similar to (2.12) for the Maxwell tensor. At least for stationary metrics it is clear that (2.12) is equivalent to a resolvent bound near zero frequencies. As it turns out, for our purposes here it is actually more efficient to work directly with a zero frequency bound, even though our metric is allowed to depend on time. We note that one could also harmlessly carry out a similar substitution in the approach in [23] for the scalar wave equation, using the appropriate zero resolvent bound as stated in [29]. By analogy, we will refer to the estimate we need as *the zero resolvent bound* for the Maxwell equation. To state it we consider the fixed time operator d^0, acting on 2-forms, which is obtained from d by eliminating the time derivatives. In other words, we define d^0 so that

$$d^0 F = dF - dt \wedge \mathcal{L}_{\partial_t} F. \tag{2.18}$$

Then we consider the fixed time system

$$d^0 F = G_1^0, \qquad d^0 * F = G_2^0. \tag{2.19}$$

Definition 2.5. *We say that the problem* (2.5) *satisfies the zero resolvent bound if on any time slice $t = t_0$ and for any $k \geq 0$, the system* (2.19) *satisfies the following estimate:*

$$\|F(t_0)\|_{\mathcal{LE}^k_{Max}} \lesssim \sum_{i=1}^{2} \|G_i^0(t_0)\|_{\mathcal{LE}^{*,k}_{Max}} \tag{2.20}$$

for all F so that the norm on the left is finite, and, in addition, the following decay condition holds at infinity:

$$\lim_{R\to\infty} \|1_{r>R} r \bar{F}(t_0)\|_{\mathcal{LE}} = 0. \tag{2.21}$$

We note that only the translation vector fields ∂ are used in (2.17) and (2.20). As part of our result, we will prove that similar bounds hold for the vector fields Ω and S. We also remark that for stationary metrics the bound (2.20) follows from the local energy decay estimates, in the same manner as in [29].

The requirement that $\bar{F}$ satisfies (2.21) is critical in order to fix the charges to zero at infinity.

3. The main result

For comparison purposes, we first state the similar result for the scalar wave equation (2.6), which was proved in [23]:

Theorem 3.1. *Let g be a metric which satisfies the conditions (i), (ii), $(iii)_a$ or (i), (ii), $(iii)_b$. Assume that weak local energy decay and stationary local energy bounds hold for solutions to the wave equation* (2.6)*. Suppose (u_0, u_1) and f are supported inside the cone $C = \{t \geq r - R_1\}$ for some $R_1 > 0$. Then for any fixed multi-index Λ the following estimate holds in normalized coordinates for a large enough m:*

$$|u^\Lambda(t,x)| \lesssim \kappa \frac{1}{\langle t\rangle \langle t-r\rangle^2}, \qquad |\nabla u^\Lambda(t,x)| \lesssim \kappa \frac{1}{\langle r\rangle \langle t-r\rangle^3} \tag{3.1}$$

where

$$\kappa = \|\nabla u(0)\|_{H^m} + \|t^{\frac{5}{2}} f^{\leq m}\|_{LE^*} + \|\langle r\rangle t^{\frac{5}{2}} \nabla f^{\leq m}\|_{LE^*}.$$

We are now ready to state the main result of the paper. Consider the frame $(\partial_u, \partial_v, e_A, e_B)$, where as usual we set

$$u = t - r, \qquad v = t + r$$

and (e_A, e_B) is an orthonormal frame of the unit sphere $(\mathbb{S}^2, d\omega)$. We have:

Theorem 3.2. *Let g be a metric which satisfies the conditions (i), (ii), $(iii)_a$ or (i), (ii), $(iii)_b$. Assume that the evolution* (2.5) *satisfies the weak local energy bounds* (2.17) *and the zero resolvent bound from Definition 2.5. Moreover, let $F(0)$ and G be supported inside the cone $C = \{t \geq r - R_1\}$ for some $R_1 > 0$, and let F solve* (2.5)*. Then the* *following peeling estimates hold in normalized coordinates for large enough m:*

$$
\begin{aligned}
|F_{uA}| &\lesssim \kappa \frac{1}{\langle t\rangle \langle t-r\rangle^3} \\
|F_{uv}| &\lesssim \kappa \frac{1}{\langle t\rangle^2 \langle t-r\rangle^2} \\
|F_{AB}| &\lesssim \kappa \frac{1}{\langle t\rangle^2 \langle t-r\rangle^2} \\
|F_{vA}| &\lesssim \kappa \frac{1}{\langle t\rangle^3 \langle t-r\rangle}
\end{aligned}
\tag{3.2}
$$

where

$$\kappa = E^m(0) + \sum_{i=1}^{2} \Big(\| t^{\frac{7}{2}} \langle r\rangle^{-1} G_i^{\leq m} \|_{LE^*} + \| t^{\frac{7}{2}} \langle r\rangle \overline{G_i}^{\leq m} \|_{LE^*} \Big).$$

Similar bounds will also hold for F^Λ for $|\Lambda| \ll m$.

It is useful at this point to review the situations where we already know that the hypothesis of the theorem is verified. So far, this is only the case for spherically symmetric black hole space-times, where we can use the result of [28]; this is further discussed at the end of this section. Another interesting case where we are almost there is that of Kerr metrics with small angular momentum. There we have available the result of [2]. Unfortunately this result only applies for solutions to the homogeneous Maxwell equation, so it cannot be applied directly. We do note that there are standard duality arguments which allow one to pass from the homogeneous to the ingomogeneous problem in local energy bounds. However, in the present situation such arguments would have to be adjusted to deal with charges.

We further remark that in a compact spatial region we obtain the rate of decay of t^{-4} for all components. This rate of decay is better than the rate of $t^{-\frac{5}{2}}$ that was obtained, for Minkowski space times, in [7]. We also note that the t^{-4} rate of decay for F_{uv}, F_{AB} on Schwarzschild space-times was previously obtained in [14, 15] by making heavy use of the stationarity and radial symmetry of the problem.

On the other hand, we note that various components (layers) of F expressed in the null frame are decaying at different rates along outgoing null cone. This type of behavior is known as peeling estimates and has been first observed in the physics literature in [27], [24]. For the Minkowski space-time, peeling estimates are known, see for example [7], and similar results have been obtained for Schwarzschild space-times, see for instance [18] and [19]. See also the related results [3], [14, 15], [16], [17] for decay estimates for Maxwell fields on Schwarzschild geometries.

Finally, we reemphasize the role played by the improved decay assumptions (precisely by an additional factor of r) on the radial part of the source term G. This guarantees that our solutions effectively behave as zero charge solutions, and the residual charge inside the cone plays only a perturbative role in the analysis. As part of our analysis, we obtain a better decay rate for the radial part of the Maxwell field, namely

$$|\bar{F}| \lesssim \kappa \frac{1}{\langle t\rangle^2 \langle r\rangle \langle t-r\rangle^2} \tag{3.3}$$

The rest of the paper is dedicated to the proof of Theorem 3.2. In Section 4 we supplement the local energy estimates (2.17) and the zero resolvent bounds (2.20), which are assumed to hold only for the translation vector fields ∂, with similar estimates involving Ω and S. Section 5 is dedicated to obtaining zero resolvent bounds with different weights at infinity. Section 6 contains an improvement on the bounds for the radial part of the tensor. Finally, Section 7 is the main part of the proof and is divided into two parts. In the first part we treat the Maxwell system as a system of wave equations and mimic the proof of the main result in [23] to get the rates of decay (3.1) for all components of the tensor F.[2] In the second part, we use the Maxwell system to improve the rates of decay and obtain the peeling estimates (3.2).

3.1. **The case of spherically symmetric metrics.** Here we provide a brief discussion of the spherically symmetric case, which is discussed in [28]. The situation considered there is that of spherically symmetric black hole space times with the following two properties:

- The event horizon is nondegenerate.
- The trapped set (photon sphere) is unique, and strictly hyperbolic.

For such space-times we have:

Proposition 3.3. *The hypothesis of Theorem 3.2 is satisfied for spherically symmetric black hole space times as in* [28].

Outline of proof. The steps of the proof are as follows:

1. Uncouple the radial and nonradial components $\bar{F}$ and $F - \bar{F}$.

2. For the radial components the equations take essentially the form

$$d(r^2\bar{F}) = r^2\bar{J},$$

see the equations (1.11) in [28]. Thus all bounds for the radial components $\bar{F}$ are obtained by direct integration from infinity, see also Remark 1.6 in [28].

3. The local energy bound (2.17) holds for the nonradial part with $k = 0$; this is the main result of [28], Theorem 1.3. In effect the result in there is more akin to (2.16) with $k = 0$, with a loss localized to the trapped set, i.e. the photon sphere.

4. The local energy bound (2.17) holds for the nonradial part with $k \geq 1$. This is by now a fairly standard argument, using the red shift property on the horizon, and only elliptic analysis away from it. This mirrors prior work of various authors for the scalar wave equation, see for instance [10], [30]. More precisely, one can add derivatives to the estimates as in the proof of Theorem 4.4 in [30].

5. The zero resolvent bound in Definition 2.5 holds. This follows by Plancherel's theorem from the $k = 0$ form of the local energy decay bound (2.17), by an argument similar to the one in [29] for the scalar wave equation. □

[2]This is where we use the estimates from Sections 5 and 6.

4. Vector field estimates.

As stated in (2.17) and (2.20), both the local energy decay and the zero resolvent bounds are assumed to hold for the derivatives of F. Our goal here is to extend these properties to the full set of vector fields Z, i.e. including the rotations Ω and scaling S, applied the Maxwell field F. The result is summarized in the following lemma:

Lemma 4.1. *Assume that weak local energy decay and the zero resolvent bound,* (2.17) *and* (2.20)*, hold. Then we also have*

$$\sup_{t>t_0} E[F^{\leq m}](t) + \|F^{\leq m}\|_{LE_{Max}} \lesssim E^1[F^{\leq m}](t_0) + \sum_{i=1}^{2} \|G_i^{\leq m+1}\|_{LE^*_{Max}}. \tag{4.1}$$

$$\|F^{\leq m}(t_0)\|_{\mathcal{LE}_{Max}} \lesssim \sum_{i=1}^{2} \|G_i^{0,\leq m}(t_0)\|_{\mathcal{LE}^*_{Max}}. \tag{4.2}$$

Proof. We begin with (4.1). Note that for any vector field X and F satisfying (2.5), we have

$$d(\mathcal{L}_X F) = \mathcal{L}_X G_1, \qquad d * (\mathcal{L}_X F) = \mathcal{L}_X G_2 + H,$$

where

$$H = d([*, \mathcal{L}_X]F). \tag{4.3}$$

We now need to commute the Lie derivative with the Hodge star. By using the well-known formulas

$$(\mathcal{L}_X F)_{\alpha\beta} = X^\gamma \partial_\gamma F_{\alpha\beta} + F_{\gamma\beta}\partial_\alpha X^\gamma + F_{\alpha\gamma}\partial_\beta X^\gamma, \tag{4.4}$$

$$(*F)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\sqrt{-g}g^{\gamma\mu}g^{\delta\nu}F_{\mu\nu}, \tag{4.5}$$

we easily obtain

$$\begin{aligned}([*, \mathcal{L}_X]F)_{\alpha\beta} = &-\frac{1}{2}X(\epsilon_{\gamma\delta\alpha\beta}\sqrt{-g}g^{\gamma\mu}g^{\delta\nu})F_{\mu\nu} + \frac{1}{2}\epsilon_{\gamma\delta\alpha\beta}\sqrt{-g}g^{\gamma\mu}g^{\delta\nu}(F_{\rho\nu}\partial_\mu X^\rho + F_{\mu\rho}\partial_\nu X^\rho)\\ &-\frac{1}{2}\sqrt{-g}g^{\gamma\mu}g^{\delta\nu}F_{\mu\nu}(\epsilon_{\gamma\delta\rho\beta}\partial_\alpha X^\rho + \epsilon_{\gamma\delta\alpha\rho}\partial_\beta X^\rho).\end{aligned} \tag{4.6}$$

If $X \in \Omega$ we obtain that

$$[*, \mathcal{L}_X]F \in S^Z(r^{-2})(F), \tag{4.7}$$

and thus also

$$H \in S^Z(r^{-3})(F) + S^Z(r^{-2})(\mathcal{L}_\partial F). \tag{4.8}$$

Here (4.7) follows from (4.6), the fact that the commutator vanishes for spherically symmetric metrics (since Ω would then be a Killing vector field) and the condition (ii) on the metric g. We note that (4.8) uses the hypothesis (2.1) but that is not strictly required for the proof of this lemma as it suffices to have r^{-2}-type decay on the first term.

Unfortunately this is not quite enough to close the argument. Indeed, we would like to prove that

$$\sup_{t>t_0} E[\mathcal{L}_X F](t) + \|\mathcal{L}_X F\|_{LE_{\text{Max}}} \lesssim E^1[F^{\leq 3}](t_0) + \sum_{i=1}^{2} \|G_i^{\leq 4}\|_{LE^*_{\text{Max}}}. \tag{4.9}$$

A first computation, using (2.17), gives

$$\sup_{t>t_0} E[\mathcal{L}_X F](t) + \|\mathcal{L}_X F\|_{LE_{\text{Max}}} \lesssim E^1[F^{\leq 3}](t_0) + \sum_{i=1}^{2} \|\mathcal{L}_X G_i^{\leq 1}\|_{LE^*_{\text{Max}}} + \|H\|_{LE^{*,1}_{\text{Max}}}$$

while (4.8) combined with (2.17) yields

$$\|H\|_{LE^{*,1}_{\text{Max}}} \lesssim E^1[F^{\leq 3}](t_0) + \sum_{i=1}^{2} \|G_i^{\leq 4}\|_{LE^*_{\text{Max}}} + \|\langle r\rangle \overline{H}\|_{LE^{*,1}}.$$

We would like to combine the last two bounds. This almost works, except for the radial part $\overline{H}$; indeed, a priori one can only estimate

$$\|\langle r\rangle \overline{H}\|_{LE^{*,1}} \lesssim \|\langle r\rangle^{-1} F\|_{LE^{*,2}}$$

The term on the right is not controlled by (2.17) (though the failure is only logarithmic). To avoid this issue, we remove this bad term by introducing a correction $\tilde{F}$ of $\mathcal{L}_X F$ as follows:

$$* \tilde{F} = (\overline{[*, \mathcal{L}_X] F})_{\phi\theta} d\omega^2 \tag{4.10}$$

Clearly by (4.7)

$$\tilde{F} \in S^Z(r^{-2}) F. \tag{4.11}$$

Thus

$$\sup_{t>t_0} E[\tilde{F}](t) + \|\tilde{F}\|_{LE_{\text{Max}}} \lesssim \sup_{t>t_0} E[F](t) + \|F\|_{LE_{\text{Max}}}$$

with room to spare, so it is enough to prove the bound

$$\sup_{t>t_0} E[\mathcal{L}_X F - \tilde{F}](t) + \|\mathcal{L}_X F - \tilde{F}\|_{LE_{\text{Max}}} \lesssim E^1[F^{\leq 3}](t_0) + \sum_{i=1}^{2} \|G_i^{\leq 4}\|_{LE^*_{\text{Max}}}$$

We have

$$d * (\mathcal{L}_X F - \tilde{F}) = \mathcal{L}_X G_2 + H - d * \tilde{F},$$

Since d annihilates $(\overline{[*, \mathcal{L}_X] F})_{tr} dt \wedge dr$ and due to our choice of $\tilde{F}$, the difference $H - d * \tilde{F}$ has no radial mode and can be estimated by

$$\|H - d * \tilde{F}\|_{LE^{*,1}_{\text{Max}}} \lesssim \|F\|_{LE^2} \lesssim E^3(t_0) + \sum_{i=1}^{2} \|G_i\|_{LE^{*,3}_{\text{Max}}}$$

On the other hand, we have

$$d(\mathcal{L}_X F - \tilde{F}) = \mathcal{L}_X G_1 - d\tilde{F}$$

so we need to bound the last term in $LE^{*,1}_{\text{Max}}$. Again, one may be concerned with the radial part. However, it is easy to see, using the asymptotic flatness of the metric, that

$$\tilde{F} - (\overline{\tilde{F}})_{tr} dt \wedge dr \in S^Z(r^{-1}) \tilde{F}$$

Hence we obtain the favorable expression

$$d\tilde{F} \in S^Z(r^{-1})\mathcal{L}_\partial \tilde{F} + S^Z(r^{-2})\tilde{F}$$

which, taking (4.11) into account, suffices in order to estimate $\|d\tilde{F}\|_{LE^{*,1}_{\text{Max}}}$ by $\|F\|_{LE^2}$. This completes the proof of (4.9).

Next we turn our attention to the scaling vector field S. With H as in (4.3), it is enough to prove that

$$H \in S^Z(r^{-2})(F, \mathcal{L}_{\{\partial,\Omega\}}F) + S^Z(r^{-1})G_2 \tag{4.12}$$

The same arguments as above will then yield the analogue of (4.9), namely

$$\sup_{t>t_0} E[\mathcal{L}_S F](t) + \|\mathcal{L}_S F\|_{LE_{\text{Max}}} \lesssim E^1[F^{\leq 9}](t_0) + \sum_{i=1}^{2} \|G_i^{\leq 10}\|_{LE^*_{\text{Max}}}. \tag{4.13}$$

We immediately get that $H \in S^Z(r^{-1})(F^{\leq 1})$ by (4.6) and the fact that S is a conformal Killing vector field for the Minkowski metric. We also note that since $g_{sr} \in S^Z(r^{-2})$, it is enough to prove (4.12) for the spherically symmetric part $\tilde{g} = m + g_{lr}$, which in normalized coordinates can be written (see (2.2)):

$$\tilde{g} = -dt^2 + dr^2 + r^2(1 + g_\omega(r))d\omega^2, \qquad g_\omega \in S^Z_{rad}(r^{-1}). \tag{4.14}$$

Note in particular that $\tilde{g}$ is diagonal in the (t, r, ϕ, θ) coordinates.

A careful inspection of (4.6) reveals that

$$\tilde{H}_{\alpha\beta} := ([*_{\tilde{g}}, \mathcal{L}_S]F)_{\alpha\beta} = \epsilon_{\gamma\delta\alpha\beta}(-S(\sqrt{-\tilde{g}}\ \tilde{g}^{\gamma\gamma}\tilde{g}^{\delta\delta}) + \kappa\sqrt{-\tilde{g}}\ \tilde{g}^{\gamma\gamma}\tilde{g}^{\delta\delta})F_{\gamma\delta} \tag{4.15}$$

where $(\alpha, \beta, \gamma, \delta)$ is some permutation of (t, r, ϕ, θ) and

$$\kappa = \begin{cases} -2 & (\alpha,\beta) = (t,r), \\ 2 & (\alpha,\beta) = (\phi,\theta), \\ 0 & \text{otherwise.} \end{cases}$$

We remark that, due to (4.14), we have

$$\tilde{H}_{t\theta} = \tilde{H}_{t\phi} = \tilde{H}_{r\theta} = \tilde{H}_{r\phi} = 0.$$

We now take the exterior derivative of the tensor $\tilde{H}$, and subsequently pass to the (t, r, A, B) frame. Every time a derivative falls on the metric coefficients, we gain a factor of r^{-2}. Since $e_{A,B} = \frac{1}{r}\Omega$, we obtain

$$H_{tAB} \approx r^{-2}\partial_t \tilde{H}_{\phi\theta} \in S^Z(r^{-1})\partial_t F_{tr} + S^Z(r^{-2})(F, \mathcal{L}_{\{\partial,\Omega\}}F)$$

$$H_{rAB} \approx r^{-2}\partial_r \tilde{H}_{\phi\theta} \in S^Z(r^{-1})\partial_r F_{tr} + S^Z(r^{-2})(F, \mathcal{L}_{\{\partial,\Omega\}}F)$$

Let us now notice that the second equation in (2.5) implies that

$$\partial_{t,r}F_{tr} \in S^Z(1)(G_2) + S^Z(r^{-1})(F, \mathcal{L}_{\{\partial,\Omega\}}F).$$

On the other hand,

$$H_{tr\phi} = \partial_\phi \tilde{H}_{tr} \in S^Z(r^{-3})\partial_\phi F_{\phi\theta}$$

and similarly for $H_{tr\theta}$. This implies

$$H_{trA}, H_{trB} \in S^Z(r^{-2})(F, \mathcal{L}_{\{\partial,\Omega\}}F).$$

Thus (4.12) for $X = S$ is now proved.

Since $\mathcal{L}_X\bar{F} = \overline{\mathcal{L}_X F}$ for $X \in \{\Omega, S\}$, (4.9) and (4.13) imply the local energy decay bound for $\mathcal{L}_\Omega F$ and for $\mathcal{L}_S F$. More derivatives can be readily added to our argument, and higher powers of Ω and S are dealt with by induction.

The proof of (4.2) is similar. Note that for any vector field X and F satisfying (2.19), we have

$$d^0(\mathcal{L}_X F) = \mathcal{L}_X G_1^0 + [\mathcal{L}_X, d - d^0]F, \qquad d^0 * (\mathcal{L}_X F) = \mathcal{L}_X G_2^0 + H + [\mathcal{L}_X, (d-d^0)*]F,$$

with H given by (4.3).

One now easily checks, using (2.18) and the fact that $\mathcal{L}_{\partial_t} F = 0$ as F is frozen at time t_0, that $[\mathcal{L}_X, d - d^0]F = 0$ if $X \in \{\Omega, S\}$. Moreover,

$$[\mathcal{L}_X, (d-d^0)*]F = dt \wedge \mathcal{L}_{\partial_t}[*, \mathcal{L}_X]F$$

where the Hodge star is frozen at time t_0.

The proofs of the analogues of (4.9) and (4.13), namely

$$\|\mathcal{L}_\Omega F(t_0)\|_{\mathcal{LE}_{\text{Max}}} \lesssim \sum_{i=1}^{2} \|G_i^{0,\le 3}(t_0)\|_{\mathcal{LE}^*_{\text{Max}}},$$

$$\|\mathcal{L}_S F(t_0)\|_{\mathcal{LE}_{\text{Max}}} \lesssim \sum_{i=1}^{2} \|G_i^{0,\le 9}(t_0)\|_{\mathcal{LE}^*_{\text{Max}}},$$

follow from using (2.20) and the same methods as above. The lemma follows by induction. □

5. Elliptic zero resolvent bounds.

The zero resolvent bound from (4.2) can be viewed more as a qualitative statement about the absence of zero eigenvalues and resonances (except for the charge induced modes, which we asymptotically identify with the radial part of F). Because of this, one has a choice over the weights that are used at infinity, very much like in the similar estimates for the inverse Laplacian. This idea is explored in this section and will play a key role in obtaining the correct pointwise decay estimates in a bounded region. Our main result is as follows:

Lemma 5.1. *The following fixed time estimates for solutions to* (2.19)*, restricted to fields $F \in \mathcal{LE}_{Max}$ with the additional property* (2.21)*, are equivalent:*

$$\|F^{\le m}(t_0)\|_{\mathcal{LE}} + \|\langle r\rangle \overline{F}^{\le m}(t_0)\|_{\mathcal{LE}} \lesssim \sum_{i=1}^{2} \|G_i^{0,\le m}(t_0)\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{G_i^0}^{\le m}(t_0)\|_{\mathcal{LE}^*}, \tag{5.1}$$

(5.2)

$$\|\langle r\rangle^{-1} F^{\le m}(t_0)\|_{\mathcal{LE}} + \|\langle r\rangle \overline{F}^{\le m}(t_0)\|_{\mathcal{LE}} \lesssim \sum_{i=1}^{2} \|\langle r\rangle^{-1} G_i^{0,\le m}(t_0)\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{G_i^0}^{\le m}(t_0)\|_{\mathcal{LE}^*}.$$

We remark that (5.1) is the same as (4.2), so it holds true under the assumption (2.20) due to Lemma 4.1.

We note that the above estimates are required to hold whenever F has the regularity stated in the beginning, and the right hand side is finite. The a priori regularity of F is needed in order to preclude the existence of solutions to the homogeneous d^0 system. We will only use these for compactly supported F, but for the proof it is more convenient to work with a weaker a priori decay assumption (2.21). We also remark that this is not a solvability property, it is just an a priori bound.

Proof. We will first prove the lemma for $m = 0$. In order to simplify the notation, since all the analysis takes place on a $\{t = t_0\}$ slice, we will drop t_0 for the rest of the proof. The weights in the two estimates are comparable in a compact set. Thus, the proof is primarily concerned with the analysis at infinity. But at infinity our problem is reasonably well approximated by the Minkowski problem, so for the most part it suffices to do a perturbative analysis. We begin with a brief analysis of what happens in the Minkowski space-time.

The Minkowski case. Denoting by $\bigstar$ the Hodge star of the Minkowski metric, the Minkowski equation has the form

$$d^0 F = G^0_1, \qquad d^0 \bigstar F = G^0_2 \tag{5.3}$$

where d^0 is now the standard exterior differentiation on a fixed time slice. We will prove both (5.1) and (5.2) in the Minkowski case by separating the radial and nonradial parts. We remark that our proof also gives the recipe for constructing the unique solution F which satisfies (2.21).

For the radial parts we have

$$\partial_r(r^2\overline{F_{AB}}) = r^2\overline{(G^0_1)}_{rAB}, \qquad \partial_r(r^2\overline{F^{\bigstar}_{AB}}) = r^2\overline{(G^0_2)}_{rAB}$$

where $F^{\bigstar} = \bigstar F$. The decay condition (2.21) at infinity allows us integrate these equations from infinity to uniquely determine the components $\overline{F_{AB}}$ and $\overline{F^{\bigstar}_{AB}}$. Outside a ball these will satisfy the straightforward bound

$$\|r(\overline{F_{AB}}, \overline{F^{\bigstar}_{AB}})\|_{\mathcal{LE}} + \|r\nabla(\overline{F_{AB}}, \overline{F^{\bigstar}_{AB}})\|_{\mathcal{LE}} \lesssim \|r(\overline{G^0_1}, \overline{G^0_2})\|_{\mathcal{LE}^*}. \tag{5.4}$$

We remark that in the Minkowski case the boundary condition at infinity will in general force an r^{-2} blow-up at zero for the radial part. In our case this does not happen because of our a-priori assumption $F \in \mathcal{LE}_{\mathrm{Max}}$.

For the nonradial part we argue in a more standard manner. For any tensor A, let $A_{nr} = A - \bar{A}$. We can rewrite (5.3) as

$$\Delta_x F_{nr,\alpha\beta} = (\bigstar d^0 \bigstar (G^0_{1,nr}) + d^0 \bigstar (G^0_{2,nr}))_{\alpha\beta} \tag{5.5}$$

where Δ_x is the usual Euclidean Laplacian. This is solved in the standard manner, using the fundamental solution for the Laplacian. Then the estimate

$$\|r^{-1}F_{nr}\|_{L^2} + \|\nabla_x F_{nr}\|_{L^2} \lesssim \sum_{i=1}^{2} \|G^0_{i,nr}\|_{L^2} \tag{5.6}$$

is a direct consequence of the direct elliptic estimate for $\nabla_x F$, coupled with Hardy's inequality to get the bound for F.

To prove either (5.1) or (5.2) it suffices to start with G_i^0 supported in a fixed dyadic region A_R.

Let us start with (5.2). Clearly (5.6) already suffices when $|x| > \frac{R}{8}$. When $|x| < \frac{R}{8}$, on the other hand, F_{nr} is harmonic, so we trivially obtain the pointwise bound

$$|F_{nr}(x)| + R|\nabla_x F_{nr}(x)| \lesssim R^{-\frac{1}{2}}(\|r^{-1}F_{nr}\|_{L^2} + \|\nabla_x F_{nr}\|_{L^2})$$

which proves (5.2) for $|x| < \frac{R}{8}$.

For (5.1), note first that (5.6) suffices when $|x| < 8R$. For $|x| > 8R$ we note that we can write (5.5) as

$$\Delta_x F_{nr,\alpha\beta}(x) = \sum_{i=1,2} \sum c_{ij\alpha\beta} \partial_j (G^0_{i,nr})_{\alpha\beta}$$

for some constants $c_{ij\alpha\beta}$. By using the fundamental solution of the Laplacian and the support property of G_i^0 we obtain the pointwise bound

$$|F_{nr}(x)| + |x||\nabla F_{nr}(x)| \lesssim \frac{R}{|x|^2} \sum_{i=1}^{2} \|G^0_{i,nr}\|_{\mathcal{LE}^*},$$

which immediately implies (5.1).

We further observe that in the Minkowski case we have actually proved a strengthened form of (5.1) and (5.2), which includes gradient bounds on the left:

$$\|F\|_{\mathcal{LE}} + \|\langle r\rangle \nabla_x F\|_{\mathcal{LE}} + \|\langle r\rangle \bar{F}\|_{\mathcal{LE}} + \|\langle r\rangle^2 \nabla_x \bar{F}\|_{\mathcal{LE}} \lesssim \|G^0\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{G^0}\|_{\mathcal{LE}^*}, \tag{5.7}$$

$$\|\langle r\rangle^{-1} F\|_{\mathcal{LE}} + \|\nabla_x F\|_{\mathcal{LE}} + \|\langle r\rangle \bar{F}\|_{\mathcal{LE}} + \|\langle r\rangle^2 \nabla_x \bar{F}\|_{\mathcal{LE}} \lesssim \|\langle r\rangle^{-1} G^0\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{G^0}\|_{\mathcal{LE}^*}. \tag{5.8}$$

By standard elliptic estimates, similar gradient terms can be added on the left in (5.1) and (5.2) in the nontrapping case. However, in the black hole case this can be done only outside a ball, more precisely in the region where ∂_t is time-like.

The general case as a perturbation of Minkowski. Starting with the equation (2.19), we write it as a perturbation of the Minkowski problem (5.3) as follows:

$$d^0 F = G_1^0, \qquad d^0 \bigstar F = G_2^0 + d^0(\bigstar - *)F. \tag{5.9}$$

In order to work with this, we need to understand the size of the terms in the last expression. Our asymptotic flatness assumptions provide the following expansion:

$$d^0(\bigstar - *)F \in S^Z(r^{-1})\nabla_x F + S^Z(r^{-2})F, \tag{5.10}$$

while for the radial part,

$$\overline{d^0(\bigstar - *)F} \in S^Z(r^{-1})\nabla_x \overline{F} + S^Z(r^{-2})(\nabla_x F + \overline{F}) + S^Z(r^{-3})F. \tag{5.11}$$

Next we use the Minkowski analysis above to deal with the general case. We need two slightly different arguments in order to go up and down in terms of decay rates.

The proof of (5.1) $\Longrightarrow$ (5.2). The main idea is to peel off the far part of the solution to (2.19) using a simple parametrix. Precisely, it suffices to construct an approximate solution $\tilde{F}$ near infinity which satisfies the bound (5.2), as well as the error estimate

$$\|d^0\tilde{F} - G_1^0\|_{\mathcal{LE}^*_{\text{Max}}} + \|d^0 * \tilde{F} - G_2^0\|_{\mathcal{LE}^*_{\text{Max}}} \lesssim RHS(5.2). \tag{5.12}$$

Then the desired bound (5.2) for F follows by applying (5.1) to $F - \chi\tilde{F}$, where χ is a smooth radial cutoff function which selects the exterior of a large ball.

The simplest idea to construct an approximate solution for (2.19) near infinity would be to treat the far away part of the equation (2.19) as a perturbation of the Laplacian. This would work in order to prove any intermediate bound between (5.1) and (5.2), but not (5.2); this is because at the level of (5.2) the radial and nonradial modes become strongly coupled. To remedy this, we solve directly for the radial parts, and perturbatively only for the nonradial components.

Precisely, the radial part of the equations (2.19) yields the equations

$$\partial_r(r^2\overline{F}_{AB}) = r^2\overline{G^0_1}_{rAB}, \qquad \partial_r(r^2\overline{F^*_{AB}}) = r^2\overline{G^0_2}_{rAB}$$

where $F^* = *F$. We integrate these equations from infinity to uniquely determine the components $\overline{F_{AB}}$ and $\overline{F^*_{AB}}$. Outside a ball these will satisfy the straightforward bound

$$\|r(\overline{F_{AB}}, \overline{F^*_{AB}})\|_{\mathcal{LE}} + \|r\nabla(\overline{F_{AB}}, \overline{F^*_{AB}})\|_{\mathcal{LE}^*} \lesssim \|r(\overline{G^0_1}, \overline{G^0_2})\|_{\mathcal{LE}^*} \tag{5.13}$$

which is akin to the Minkowski bound (5.4).

To define $\tilde{F}$, we first obtain its nonradial part $\tilde{F}_{nr}$ by solving the Minkowski space-time version (5.3) of our equations. As discussed above, this satisfies the bounds

$$\|r^{-1}\tilde{F}_{nr}\|_{\mathcal{LE}} + \|\nabla\tilde{F}_{nr}\|_{\mathcal{LE}} \lesssim \sum_{i=1}^{2} \|\langle r\rangle^{-1} G^0_{i,nr}\|_{\mathcal{LE}^*}. \tag{5.14}$$

Now we define the radial part of $\tilde{F}$ by requiring it to match the two radial components of F directly computed above, namely

$$\overline{\tilde{F}_{AB}} = \overline{F_{AB}}, \qquad \overline{\tilde{F}^*_{AB}} = \overline{F^*_{AB}}.$$

The first equation gives directly $\tilde{F}_{AB}$. From the second we compute

$$\overline{\tilde{F}_{tr}} \in S^Z_{rad}(1)\overline{F^*_{AB}} + S_Z(r^{-2})\tilde{F}_{nr}. \tag{5.15}$$

Our construction above yields a field $\tilde{F}$ outside a large ball. By (5.13), (5.14) and (5.15) it follows that $\tilde{F}$ satisfies the bound (5.2). Further, $\tilde{F}$ solves exactly the first equation in (2.19), as well as the radial component of the second equation in (2.19). It remains to estimate the nonradial error in the second equation. Using the asymptotic flatness of the metric, we see that this is given by

$$\begin{aligned} d^0(\tilde{F}^*)_{nr} - G^0_{2,nr} &= (d^0 * \tilde{F}_{nr} + d^0 * \bar{\tilde{F}})_{nr} - G^0_{2,nr} \\ &= (d^0(* - \bigstar)\tilde{F}_{nr})_{nr} \\ &\in (S^Z(r^{-1})\nabla\tilde{F}_{nr} + S^Z(r^{-2})\tilde{F}_{nr})_{nr}. \end{aligned}$$

We can bound this error using (5.14) to obtain

$$\|r(d^0(\tilde{F}^*)_{nr} - G^0_{2,nr})\|_{\mathcal{LE}} \lesssim \sum_{i=1}^{2} \|\langle r\rangle^{-1} G^0_i\|_{\mathcal{LE}^*}.$$

This almost gives (5.12), up to a logarithmic divergence. However, our error $\tilde{G}^0_2 := G^0_2 - d * \tilde{F}$ decays better than G^0_2 by a power of r, so in order to obtain (5.12) it suffices to reiterate once more the above construction.

The proof of (5.2) $\implies$ (5.1). We begin with the series of inequalities

$$LHS((5.2)) \lesssim RHS((5.2)) \lesssim RHS((5.1)).$$

As observed earlier, we can also obtain an elliptic bound for ∇F outside a compact set. These estimates provide a weaker bound, which nevertheless suffices within a compact set. Hence a straightforward localization argument, namely replacing F with χF, allows us to reduce the problem to the case when F is supported in an exterior region $\{r \gtrsim R_2\}$.

But in this region we can replace g by m perturbatively. We write the equation (2.19) as in (5.9), and apply the bound (5.7) in the Minkowski setting. It remains to estimate the error $\|d^0(* - \bigstar)F\|_{\mathcal{LE}^*_{\mathrm{Max}}}$, for which we use the expressions (5.10) and (5.11). We obtain

$$\begin{aligned}\|d^0(* - \bigstar)F\|_{\mathcal{LE}^*_{\mathrm{Max}}} &\lesssim \|\langle r\rangle^{-1}\nabla_x F\|_{\mathcal{LE}^*} + \|\nabla_x \bar{F}\|_{\mathcal{LE}^*} + \|\langle r\rangle^{-2} F\|_{\mathcal{LE}^*_{\mathrm{Max}}} \\ &\lesssim R_2^{-1/2}(\|\langle r\rangle \nabla_x F\|_{\mathcal{LE}} + \|\langle r\rangle^2 \nabla_x \bar{F}\|_{\mathcal{LE}} + \|F\|_{\mathcal{LE}_{\mathrm{Max}}}).\end{aligned}$$

If R_2 is large enough then this term is perturbative in (5.7), and the proof of (5.1) is concluded.

This concludes the proof for $m = 0$. Higher spatial derivatives are easily introduced in the argument in an elliptic fashion. Finally, the same arguments as in Lemma 4.1 apply for Ω and S. □

6. Charges and bounds for the radial part

As explained earlier, the radial part of the solution is a good approximation of the charge near spatial infinity. In particular, we expect it to have better bounds (assuming the sources G_1 and G_2 have good decay at infinity), and we also expect it to not propagate in a dispersive fashion along the cone. However, there is some degree of freedom in our choice of coordinates, and thus in what we call the radial part. Hence, within our setup, there is some degree of mixing between radial and nonradial. The next result shows that the nonradial effects on the radial part have size r^{-2}; thus, as expected, they are weaker near infinity and stronger in a compact set. This is in a nutshell the content of the next lemma, which will come in very handily when we seek to propagate bounds for the radial part inside the cone, without any crossing penalty.

Lemma 6.1. *The radial part of F satisfies the improved estimate*

$$\|\langle r\rangle^{\frac{3}{2}} \bar{F}^{\le m}(t_0)\|_{L^2(A_R)} \lesssim \sum_{i=1}^{2} \|\langle r\rangle^2 \bar{G}_i^{\le m}(t_0)\|_{\mathcal{LE}^*} + \|\langle r\rangle^{-\frac{1}{2}} F^{\le m}(t_0)\|_{L^2(A_R)}. \tag{6.1}$$

Proof. The estimate is obvious when $R \approx 1$. When $R \gg 1$, we will use the original system (2.5), which in particular implies that

$$\partial_r(r^2\overline{F}_{AB}) = r^2\overline{(G_1)}_{rAB}, \qquad \partial_r(r^2(\overline{*F})_{AB}) = r^2\overline{(G_2)}_{rAB}. \tag{6.2}$$

Moreover, due to (4.5) we have

$$(\overline{*F})_{AB} = (1 + S^Z_{rad}(r^{-1}))\overline{F}_{tr} + S^Z(r^{-2})F, \tag{6.3}$$

as well as

$$\partial_r(\overline{*F})_{AB} = (1 + S^Z_{rad}(r^{-1}))\partial_r\overline{F}_{tr} + S^Z_{rad}(r^{-2})\overline{F}_{tr} + S^Z(r^{-2})\partial_r F + S^Z(r^{-3})F. \tag{6.4}$$

After integrating from infinity and applying the Schwarz inequality for the terms involving G_i, we obtain the desired conclusion for $m = 0$:

$$\|r^{\frac{3}{2}}\bar{F}(t_0)\|_{L^2(A_R)} \lesssim \sum_{i=1}^{2} \|r^2\bar{G}_i(t_0)\|_{\mathcal{LE}^*} + \|r^{-\frac{1}{2}}F(t_0)\|_{L^2(A_R)}. \tag{6.5}$$

We now need to commute with vector fields in Z. After commuting (6.2) with ∂_t and ∂_r we easily obtain that

$$\|r^{\frac{3}{2}}\partial_{t,r}\bar{F}(t_0)\|_{L^2(A_R)} \lesssim \sum_{i=1}^{2} \|r^2\bar{G}_i^{\leq 1}(t_0)\|_{\mathcal{LE}^*} + \|r^{-\frac{1}{2}}F^{\leq 1}(t_0)\|_{L^2(A_R)}.$$

Since $\partial_{x_i} f(t, r) \in S^Z(1)\partial_r f(t, r)$ for any radially symmetric function f, the inequality above also holds for all derivatives.

On the other hand, for a vector field $X \in \{\Omega, S\}$ we know that $\mathcal{L}_X\bar{F} = \overline{\mathcal{L}_X F}$. After applying $\mathcal{L}_X$ to (2.5) we get

$$\partial_r(r^2(\mathcal{L}_X\bar{F})_{AB}) = r^2(\mathcal{L}_X\bar{G}_1)_{rAB}, \qquad \partial_r(r^2(\overline{*\mathcal{L}_X F})_{AB} + r^2(\overline{[\mathcal{L}_X, *]F})_{AB}) = r^2(\mathcal{L}_X\bar{G}_2)_{rAB}.$$

Due to (4.5) we have

$$(\overline{*\mathcal{L}_X F})_{AB} = (1 + S^Z_{rad}(r^{-1}))(\mathcal{L}_X\bar{F})_{tr} + S^Z(r^{-2})(\mathcal{L}_X F).$$

Thus after integrating from infinity and applying Hölder's inequality for the terms involving G_i, we obtain

$$\|r^{\frac{3}{2}}\mathcal{L}_X\bar{F}(t_0)\|_{L^2(A_R)} \lesssim \sum_{i=1}^{2} \|r^2\mathcal{L}_X G_i(t_0)\|_{\mathcal{LE}^*} + \|r^{-\frac{1}{2}}\mathcal{L}_X F(t_0)\|_{L^2(A_R)} + \|r^{\frac{3}{2}}(\overline{[\mathcal{L}_X, *]F})_{AB}\|_{L^2(A_R)}.$$

When $X \in \Omega$ we see from (4.6) and (4.7) that

$$|[\mathcal{L}_X, *]F| \lesssim r^{-2}|F|.$$

On the other hand, by using (4.15) and the fact that $g_{sr} \in S^Z(r^{-2})$, we see that

$$|(\overline{[\mathcal{L}_S, *]F})_{AB}| \lesssim r^{-1}|\overline{F}| + r^{-2}|F|.$$

We thus obtain in both cases

$$\|r^{\frac{3}{2}}\mathcal{L}_X\bar{F}(t_0)\|_{L^2(A_R)} \lesssim \sum_{i=1}^{2} \|r^2\mathcal{L}_X G_i(t_0)\|_{\mathcal{LE}^*} + \|r^{-\frac{1}{2}}\mathcal{L}_X F(t_0)\|_{L^2(A_R)}.$$

We can now use induction to conclude that (6.1) holds for all m. ☐

7. Proof of the main result

The proof of the main theorem will be divided into two parts. We first mimic the approach used in [23] to prove Theorem 3.1 to obtain pointwise bounds which are similar to those in the scalar case:

$$|F^{\leq n}_{\alpha\beta}| \lesssim \frac{\kappa_1}{\langle t\rangle\langle t-r\rangle^2}, \qquad |\nabla F^{\leq n}_{\alpha\beta}| \lesssim \frac{\kappa_1}{\langle r\rangle\langle t-r\rangle^3} \tag{7.1}$$

where

$$\kappa_1 = E^{n+m}(0) + \sum_{i=1}^{2} \|t^{\frac{5}{2}} G_i^{\leq n+m}\|_{LE^*} + \|t^{\frac{5}{2}} r\overline{G_i}^{\leq n+m}\|_{LE^*}.$$

We then use (7.1) combined with the Maxwell system to improve the decay near the cone to the peeling estimates (3.2).

7.1. **The Maxwell system as a wave equation.** We start by rewriting the Maxwell system as a system of wave equations for each component. We have

$$\nabla_{[\alpha} F_{\beta\gamma]} = G_{1\alpha\beta\gamma}, \qquad \nabla^\alpha F_{\alpha\beta} = - * G_{2\beta}. \tag{7.2}$$

Differentiating the first equation we get

$$\nabla^\alpha\nabla_\alpha F_{\beta\gamma} + [\nabla^\alpha, \nabla_\gamma] F_{\alpha\beta} + [\nabla^\alpha, \nabla_\beta] F_{\gamma\alpha} \ = \nabla^\alpha G_{1\alpha\beta\gamma} - \nabla_{[\beta} * G_{2\gamma]} \ ,$$

where we have used $\nabla^\alpha F_{\alpha\beta} = - * G_{2\beta}$ in the second and third term. The commutators are curvature contributions, and cleaning these up we get:

$$\Box_g F_{\alpha\beta} - R_\alpha{}^\gamma F_{\gamma\beta} - R_\beta{}^\gamma F_{\alpha\gamma} + R_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} \ = \ \nabla^\gamma G_{1\gamma\alpha\beta} - \nabla_{[\alpha} * G_{2\beta]} \ . \tag{7.3}$$

Here $\Box_g$ is the covariant wave equation acting on two forms; we would like to replace this with $\Box_g[F_{\alpha\beta}]$, the scalar d'Alembertian applied to each component separately. For that we compute

$$\begin{aligned} \nabla^\gamma\nabla_\gamma F_{\alpha\beta} = \nabla^\gamma \partial_\gamma [F_{\alpha\beta}] - g^{\gamma\delta}\Big[&\Gamma^\sigma_{\gamma\alpha,\delta} F_{\sigma\beta} - \Gamma^\sigma_{\gamma\beta,\delta} F_{\alpha\sigma} - 4\Gamma^\sigma_{\delta[\beta} F_{\alpha]\sigma,\gamma} \\ &+ 2\Gamma^\sigma_{\delta\gamma}\Gamma^\mu_{\sigma[\beta} F_{\alpha]\mu} + 2\Gamma^\mu_{\gamma\sigma}\Gamma^\sigma_{\delta[\beta} F_{\alpha]\mu}\Big]. \end{aligned} \tag{7.4}$$

Taking into account that g is in normalized coordinates, as well as (2.1) (which in particular imply that $\Gamma^\gamma_{\alpha\beta} \in S^Z(r^{-2})$ and $R_{\alpha\beta\gamma\delta} \in S^Z(r^{-3})$), one easily obtains that each component $F_{\alpha\beta}$ satisfies

$$\Box_g [F_{\alpha\beta}] = Q_{\alpha\beta} \in S^Z(1)(G_1^{\leq 1}, G_2^{\leq 1}) + S^Z(r^{-2})\partial F + S^Z(r^{-3}) F. \tag{7.5}$$

The decay on the last term follows from (2.1), and it is here that this hypothesis is crucial.

This immediately implies the analogous equation for $\Box F_{\alpha\beta}$. After commuting with the vector fields in Z we also obtain by induction for all multi-indices Λ:

$$\Box F^\Lambda_{\alpha\beta} \in S^Z(1)(G_1^{\leq|\Lambda|+m}, G_2^{\leq|\Lambda|+m}) + S^Z(r^{-2})(\partial F^{\leq|\Lambda|+m}) + S^Z(r^{-3})(F^{\leq|\Lambda|+m}). \tag{7.6}$$

Here and in the sequel m will be a large enough number that could change from equation to equation.

We can now apply Lemma 3.10 from [23] which gives a first pointwise estimate in terms of the local energy norms:

$$|F^{\Lambda}_{\alpha\beta}| \lesssim \frac{\log\langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^{\frac{1}{2}}}\Big(\sum_{\alpha,\beta}\|F^{\leq|\Lambda|+m}_{\alpha\beta}\|_{LE^1} + \sum_{i=1}^{2}\|\langle r\rangle G_i^{\leq|\Lambda|+m}\|_{LE^*}\Big). \tag{7.7}$$

At this point of the proof we would like to analyze what happens inside the cone (the region $r \ll t$) and near the cone (the region $t \approx r$) separately. In the first region we will use the zero resolvent bound for the Maxwell system, while in the second region we will fall back onto the wave equation analysis and use the fundamental solution for the Minkowski wave equation.

7.2. **Notations and localized Klainerman-Sobolev bounds.** We first recall some notation from [23]. For the forward cone $C = \{r \leq t + R_1\}$ we consider a dyadic decomposition in time into sets

$$C_T = \{T \leq t \leq 2T, \quad r \leq t + R_1\}.$$

For each C_T we need a further double dyadic decomposition of it with respect to either the size of $t-r$ or the size of r, depending on whether we are close or far from the cone,

$$C_T = \bigcup_{1\leq R\leq T/2} C_T^R \cup \bigcup_{1\leq U<T/2} C_T^U$$

where for $R, U > 1$ we set

$$C_T^R = C_T \cap \{R < r < 2R\}, \qquad C_T^U = C_T \cap \{U < t-r < 2U\}$$

while for $R = 1$ and $U = 1$ we have

$$C_T^{R=1} = C_T \cap \{R_0 < r < D\}, \qquad D \gg R_0$$

$$C_T^{U=1} = C_T \cap \{-R_1 < t-r < 2\}.$$

By $\tilde{C}_T^R$ and $\tilde{C}_T^U$ we denote enlargements of these sets in both space and time on their respective scales. We also define

$$C_T^{<T/2} = \bigcup_{R<T/2} C_T^R,$$

while $\tilde{C}_T^{<T/2}$ is a corresponding enlargement. Finally, we will use the notation $C_T^{<T/2}(t_0) = C_T^{<T/2} \cap \{t = t_0\}$ and similarly for $\tilde{C}_T^{<T/2}(t_0)$.

The following Sobolev embeddings hold (see Lemma 3.8 from [23] for proof):

$$\|F_{\alpha\beta}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{T^{\frac{1}{2}}R^{\frac{3}{2}}}\|F^{\leq 2}_{\alpha\beta}\|_{L^2(\tilde{C}_T^R)} + \frac{1}{T^{\frac{1}{2}}R^{\frac{1}{2}}}\|\nabla F^{\leq 2}_{\alpha\beta}\|_{L^2(\tilde{C}_T^R)}, \tag{7.8}$$

respectively

$$\|F_{\alpha\beta}\|_{L^\infty(C_T^U)} \lesssim \frac{1}{T^{\frac{3}{2}}U^{\frac{1}{2}}}\|F^{\leq 2}_{\alpha\beta}\|_{L^2(\tilde{C}_T^U)} + \frac{U^{\frac{1}{2}}}{T^{\frac{3}{2}}}\|\nabla F^{\leq 2}_{\alpha\beta}\|_{L^2(\tilde{C}_T^U)}. \tag{7.9}$$

We can now use (7.8) and (7.9) to improve the decay of the derivative by a factor of r^{-1} away from the cone, respectively by a factor of $\langle t-r\rangle^{-1}$ near the cone. We proved a similar type of result for the wave equation in [23], see Proposition 3.16, though the proof in that case was somewhat different.

Let us first consider the derivatives of F in the region C_T^R. Clearly we have the bound $|e_{A,B}F^{\Lambda}_{\alpha\beta}| \lesssim \frac{1}{r}|F^{\Lambda+3}_{\alpha\beta}|$. For the time derivative, we use the fact that

$$\partial_t F^{\Lambda}_{\alpha\beta} = \frac{1}{t}SF^{\Lambda}_{\alpha\beta} - \frac{r}{t}\partial_r F^{\Lambda}_{\alpha\beta}.$$

On the other hand, the Maxwell system gives us that

$$\partial_r F^{\Lambda}_{\alpha\beta} - \delta\partial_t F^{\Lambda}_{\tilde\alpha\tilde\beta} \in S^Z(1)(G_1^{\leq|\Lambda|+m}, G_2^{\leq|\Lambda|+m}) + S^Z(r^{-1})F^{\leq|\Lambda|+m}$$

for suitable $\tilde\alpha$ and $\tilde\beta$, where $\delta = \pm 1$ if α or β equals t or r and 0 otherwise. Combining the last two relations we immediately get that

$$|\partial_{t,r}F^{\Lambda}_{\alpha\beta}| \lesssim \sum_{i=1}^{2}|G_i^{\leq|\Lambda|+m}| + r^{-1}|F^{\leq|\Lambda|+m}| \tag{7.10}$$

After applying the Sobolev embeddings (7.8) to G_i we obtain that

$$R\|\nabla F^{\Lambda}_{\alpha\beta}\|_{L^\infty(C_T^R)} \lesssim \|F^{\leq|\Lambda|+m}\|_{L^\infty(C_T^R)} + \sum_{i=1}^{2} T^{-\frac12}R^{\frac12}\|G_i^{\leq|\Lambda|+m}\|_{L^2(\tilde C_T^R)}. \tag{7.11}$$

A similar argument applied in the region C_T^U yields

$$U\|\nabla F^{\Lambda}_{\alpha\beta}\|_{L^\infty(C_T^U)} \lesssim \ \|F^{\leq|\Lambda|+m}\|_{L^\infty(C_T^U)} + T^{-\frac12}U^{\frac12}\sum_{i=1}^{2}\|G_i^{\leq|\Lambda|+m}\|_{L^2(\tilde C_T^U)}. \tag{7.12}$$

7.3. **Improved bounds in the interior.** We will now obtain improved bounds for $F^{\Lambda}_{\alpha\beta}$ and the gradient $\nabla F^{\Lambda}_{\alpha\beta}$ in the interior region $C_T^{<T/2}$. We remark that the results in [23], namely Proposition 3.14 and Proposition 3.15, which allow us to replace the factor of $\langle r\rangle$ by a factor of $\langle t\rangle$ in the right hand side of (7.7), do not directly apply in the case of the Maxwell system. Indeed, it is not clear why the stationary local energy decay (2.12) would hold for F solving the system (7.5), even if we assume that it holds for the corresponding scalar equation. Moreover, we also need to deal with the presence of the radial part of F, which decays at a different rate from the nonradial part. Instead, we will be using the zero resolvent bound (4.2) and Lemmas 5.1 and 6.1 to prove the following result which is similar to Proposition 3.15 in [23].

Proposition 7.1. *Assume that the solution to* (2.5) *satisfies the zero resolvent bound* (2.20) *for all* $T \leq t_0 \leq 2T$. *Then for* m *large enough and any multiindex* Λ *the following estimates hold:*

$$\|\langle r\rangle^{-1}F^{\Lambda}_{\alpha\beta}\|_{LE(C_T^{<\frac{T}{2}})} + \|\nabla F^{\Lambda}_{\alpha\beta}\|_{LE(C_T^{<\frac{T}{2}})} \lesssim M \tag{7.13}$$

and

$$\|F^{\Lambda}_{\alpha\beta}\|_{L^\infty(C_T^{<\frac{T}{2}})} + \|\langle r\rangle\nabla F^{\Lambda}_{\alpha\beta}\|_{L^\infty(C_T^{<\frac{T}{2}})} \lesssim \tilde M \tag{7.14}$$

where

$$M = T^{-1}\|F^{\leq|\Lambda|+m}\|_{LE(\tilde C_T^{<T/2})} + \sum_{i=1}^{2}\left(\|\langle r\rangle^{-1}G_i^{\leq|\Lambda|+m}\|_{LE^*(C_T)} + \|\langle r\rangle\bar G_i^{\leq|\Lambda|+m}\|_{LE^*(C_T)}\right),$$

$$\tilde{M} = T^{-\frac{1}{2}} M + \sup_{R<T/2} \sum_{i=1}^{2} T^{-\frac{1}{2}} R^{\frac{1}{2}} \| G_i^{\leq |\Lambda|+m} \|_{L^2(\tilde{C}_T^R)}.$$

Proof. The main estimate here is the local energy bound (7.13). Indeed, (7.14) follows from (7.13) via the Klainerman-Sobolev type bounds (7.8) and (7.11) applied successively in all dyadic regions $R < \frac{T}{2}$. We remark that (7.13) is the analogue of Proposition 3.14 in [23].

To prove (7.13), we first note that, in view of Lemma 6.1, we can freely add to M the corresponding bound for the radial part,

$$M := M + T^{-1} \| \langle r \rangle^2 \bar{F}^{\leq |\Lambda|+m} \|_{LE(\tilde{C}_T^{<T/2})}.$$

The next step is to localize the problem to C_T. Let $\chi_T(t,r)$ be a nonnegative smooth cutoff supported in $\tilde{C}_T^{<T/2}$ so that $\chi_T \equiv 1$ in $C_T^{<T/2}$. We replace F with the tensor $\tilde{F} = \chi_T F$. We see that $\tilde{F}$ satisfies the system

$$d\tilde{F} = \tilde{G}_1 := \chi_T G_1 + d\chi_T \wedge F, \qquad d * \tilde{F} = \tilde{G}_2 := \chi_T G_2 + d\chi_T \wedge *F.$$

Clearly $\nabla \chi_T$ is supported in $\tilde{C}_T^{<T/2} \setminus C_T^{<T/2}$ and the cutoff can be chosen so that $|\nabla \chi_T| \lesssim T^{-1}$. We thus obtain that

$$\begin{aligned} \| \langle r \rangle^{-1} d\chi_T \wedge F \|_{LE^*(C_T)} + \| \langle r \rangle \overline{d\chi_T \wedge F} \|_{LE^*(C_T)} + \| \langle r \rangle \overline{d\chi_T \wedge *F} \|_{LE^*(C_T)} \\ \lesssim T^{-1} \| F \|_{LE(\tilde{C}_T^{<T/2})} + T^{-1} \| \langle r \rangle^2 \overline{F} \|_{LE(\tilde{C}_T^{<T/2})} \end{aligned}$$

where for the last term on the LHS we used that

$$\overline{*F} \in S^Z(1)(\bar{F}) + S^Z(r^{-2}) F$$

After taking Lie derivatives, it is now easy to see that $\tilde{G}_i$ satisfies the inequality

$$\| \langle r \rangle^{-1} \tilde{G}_i^{\leq \Lambda+m} \|_{LE^*(C_T)} + \| \langle r \rangle \bar{\tilde{G}}_i^{\leq \Lambda+m} \|_{LE^*(C_T)} \lesssim M.$$

Thus, from here on we assume that F is (spatially) supported in $C_T^{<T/2}$ and drop the $\tilde{F}$ notation.

The next step is to see that we can further replace M by

$$M := M + T^{-1} (\| \langle r \rangle \partial_r F^{\leq |\Lambda|+m} \|_{LE(\tilde{C}_T^{<T/2})} + \| \langle r \rangle^3 \partial_r \bar{F}^{\leq |\Lambda|+m} \|_{LE(\tilde{C}_T^{<T/2})}).$$

Indeed, the first term on the right is estimated in terms of M using the pointwise bound (7.10), and for the second we use the relations (6.3)-(6.4).

We now introduce the quantities

$$\gamma^{|\Lambda|} = \sum_{i=1}^{2} \| \langle r \rangle^{-1} G_i^{\leq |\Lambda|} \|_{LE^*(C_T)}, \qquad \bar{\gamma}^{|\Lambda|} = \sum_{i=1}^{2} \| \langle r \rangle \bar{G}_i^{\leq |\Lambda|} \|_{LE^*(C_T)}$$

respectively, with $h \in [0,1]$,

$$\phi^{|\Lambda|,h} = T^h (\| \langle r \rangle^{-h} F^{\leq |\Lambda|} \|_{LE(C_T)} + \| \langle r \rangle^{1-h} \nabla_x F^{\leq |\Lambda|} \|_{LE(C_T)}),$$

$$\bar{\phi}^{|\Lambda|,h} = T^h (\| \langle r \rangle^{2-h} \bar{F}^{\leq |\Lambda|} \|_{LE(C_T)} + \| \langle r \rangle^{3-h} \nabla_x \bar{F}^{\leq |\Lambda|} \|_{LE(C_T)}).$$

With these notations, the bound to prove becomes

$$\phi^{|\Lambda|,1} + \bar{\phi}^{|\Lambda|,1} \lesssim \phi^{|\Lambda|+m,0} + \bar{\phi}^{|\Lambda|+m,0} + T\gamma^{|\Lambda|+m} + T\bar{\gamma}^{|\Lambda|+m}. \tag{7.15}$$

Indeed, the time derivatives can be easily estimated afterwards by using either the Maxwell system or the scaling vector field S.

In order to use the bounds in Lemma 5.1 we need to convert the Maxwell system (2.5) into the d^0 system (2.19) and estimate the source terms G_i^0. We will show that

(7.16)

$$G_i^{0,\Lambda} \in S^Z(1)G_i^\Lambda + \frac{1}{t}dt \wedge S^Z(1)(F^{\leq|\Lambda|+m}, r\partial_r F^{\leq|\Lambda|+m})$$

$$\overline{G_i^0}^\Lambda \in S^Z(1)\overline{G_i}^\Lambda + \frac{1}{t}dt \wedge \left[S^Z(1)(\bar{F}^{\leq|\Lambda|+m}, r\partial_r \bar{F}^{\leq|\Lambda|+m}) + S^Z(r^{-2})(F^{\leq|\Lambda|+m}, r\partial_r F^{\leq|\Lambda|+m})\right]$$

For this we use the scaling field S as a proxy for ∂_t to compute via (2.18):

(7.17)

$$G_1^0(t) = G_1(t) - dt \wedge \mathcal{L}_{\partial_t} F = G_1(t) - \frac{1}{t}dt \wedge (\mathcal{L}_S F - r\mathcal{L}_{\partial_r} F - F_{r\phi} d\phi \wedge dr - F_{r\theta} d\theta \wedge dr)$$

$$\overline{G_1^0}(t) = \overline{G_1}(t) - dt \wedge \mathcal{L}_{\partial_t} \bar{F} = \overline{G_1}(t) - \frac{1}{t}dt \wedge (\mathcal{L}_S \bar{F} - r\mathcal{L}_{\partial_r} \bar{F})$$

We also need to take Lie derivatives in (7.17). This is done using

$$\mathcal{L}_X G_1^0(t) = \mathcal{L}_X G_1(t) - dt \wedge \mathcal{L}_{\partial_t} F^{\leq m}, \qquad \mathcal{L}_X \overline{G_1^0}(t) = \mathcal{L}_X \overline{G_1}(t) - dt \wedge \mathcal{L}_{\partial_t} \bar{F}^{\leq m},$$

This is immediate for $X \in \{\partial, \Omega\}$ and a simple computation for $X = S$. The desired result (7.16) for G_1 follows by induction.

The proof of (7.16) for G_2 follows by applying the arguments above to $*F$ instead of F and using the fact that

$$\overline{*F} \in S^Z(1)\bar{F} + S^Z(r^{-2})F$$

We will now bound the sources G_i^0 and their Lie derivatives in C_T. We begin with the nonradial part, for which we have

(7.18)

$$\begin{aligned}\|\langle r\rangle^{-1} G_i^{0,\leq|\Lambda|}\|_{LE^*(C_T)} &\lesssim \|\langle r\rangle^{-1} G_i^{\leq|\Lambda|}\|_{LE^*(C_T)} \\ &\quad + \frac{1}{T}\Big(\|\langle r\rangle^{-1} F^{\leq|\Lambda|+m}\|_{LE^*(C_T)} + \|\partial_r F^{\leq|\Lambda|}\|_{LE^*(C_T)}\Big) \\ &\lesssim \gamma^{|\Lambda|} + T^{-1}\phi^{|\Lambda|+m,\frac{1}{2}}.\end{aligned}$$

On the other hand for the radial part we have

$$\begin{aligned}\|\langle r\rangle \overline{G_i^0}^{\leq|\Lambda|}\|_{LE^*(C_T)} &\lesssim \|\langle r\rangle \bar{G}_i^{\leq|\Lambda|}\|_{LE^*(C_T)} \\ &\quad + \frac{1}{T}\Big(\|\langle r\rangle \bar{F}^{\leq|\Lambda|+m}\|_{LE^*(C_T)} + \|\langle r\rangle^2 \partial_r \bar{F}^{\leq|\Lambda|+m}\|_{LE^*(C_T)} \\ &\qquad + \|\langle r\rangle^{-1} F^{\leq|\Lambda|+m}\|_{LE^*(C_T)} + \|\partial_r F^{\leq|\Lambda|+m}\|_{LE^*(C_T)}\Big) \\ &\lesssim \bar{\gamma}^{|\Lambda|} + T^{-1}\bar{\phi}^{|\Lambda|+m,\frac{1}{2}} + T^{-1}\phi^{|\Lambda|+m,\frac{1}{2}}.\end{aligned} \tag{7.19}$$

By interpolation we have bounds of the type

$$\phi^{|\Lambda|+m,\frac{1}{2}} \lesssim (\phi^{|\Lambda|,1}\phi^{|\Lambda|+2m,0})^{\frac{1}{2}}.$$

Viewed in polar self-similar coordinates in C_T^R, these interpolation bounds are nothing but standard Sobolev bounds in a unit cube. Using the interpolation estimates, from (7.18) and (7.19) we have

$$\|\langle r\rangle^{-1}G_i^{0\leq|\Lambda|}\|_{LE^*(C_T)} + \|\langle r\rangle\overline{G_i^0}^{\leq|\Lambda|}\|_{LE^*(C_T)} \lesssim \gamma^{|\Lambda|} + \bar\gamma^{|\Lambda|} + T^{-1}(\phi^{|\Lambda|,1}\phi^{|\Lambda|+2m,0})^{\frac{1}{2}} + T^{-1}(\bar\phi^{|\Lambda|,1}\bar\phi^{|\Lambda|+2m,0})^{\frac{1}{2}}.$$

Now we can apply the zero resolvent bound (5.2). Using also the bounds (7.10) to estimate $\partial_r F^{\leq|\Lambda|}$, as well as (6.3)-(6.4) to estimate $\partial_r \bar F^{\leq|\Lambda|}$, we obtain

$$\phi^{|\Lambda|,1} + \bar\phi^{|\Lambda|,1} \lesssim T\gamma^{|\Lambda|} + T\bar\gamma^{|\Lambda|} + (\phi^{|\Lambda|,1}\phi^{|\Lambda|+2m,0})^{\frac{1}{2}} + (\bar\phi^{|\Lambda|,1}\bar\phi^{|\Lambda|+2m,0})^{\frac{1}{2}}.$$

Then the desired estimate (7.15) follows by Cauchy-Schwarz. □

We will now prove (7.1) in the same way as in [23], by a bootstrap procedure. The starting point is the pointwise bound (7.7). This can be improved by replacing the r^{-1} factor by a t^{-1} factor, and complemented by a better bound for the derivative near the cone. Indeed, by using (7.7) and Hölder's inequality in (7.14) we obtain

$$|F^\Lambda_{\alpha\beta}| \lesssim C_1 \frac{\log\langle t-r\rangle}{t\langle t-r\rangle^{\frac{1}{2}}},$$

whereas using (7.7) in (7.12) yields

$$|\nabla F^\Lambda_{\alpha\beta}| \lesssim C_1 \frac{\log\langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^{\frac{3}{2}}}.$$

Here

$$C_1 = \|F^{\leq|\Lambda|+m}\|_{LE} + \sum_{i=1}^{2}\sup_{R,U} R^{\frac{1}{2}}T^{\frac{1}{2}}U^{\frac{1}{2}}\|G_i^{\leq|\Lambda|+m}\|_{L^2(C_T^{R,U})} + T\|r\bar G_i^{\leq|\Lambda|+m}\|_{LE^*(C_T)}.$$

where, following [23], $C_T^{R,U}$ stands for either C_T^R or C_T^U, with the convention that $R\approx T$ in C_T^U and $U\approx T$ in C_T^R.

7.4. **Uniform pointwise bounds.** We can now use the improved estimates in a bootstrap procedure similar to the one in [23]. The first step is to note that for a solution to the Minkowski wave equation

$$\Box w = f, \qquad w(0) = \partial_t w(0) = 0$$

so that f is supported in the forward cone $\{t > |r|\}$ we can estimate

$$|w| \lesssim \frac{1}{r}\int_{D_{tr}}\int_{\mathbb{S}^2} \rho|f^{\leq m}|d\omega ds d\rho \tag{7.20}$$

where D_{tr} is the rectangle

$$D_{tr} = \{0 \leq s-\rho \leq t-r, \quad t-r \leq s+\rho \leq t+r\}.$$

We call this computation the one dimensional reduction; this is fairly standard, and it is explained in detail in [23]. By using the above estimate in conjunction with (7.6) we improve our estimate near the cone to

$$|F^{\Lambda}_{\alpha\beta}| \lesssim C_2 \frac{\log\langle t-r\rangle}{\langle r\rangle\langle t-r\rangle}$$

where

$$C_2 = \|F^{\le|\Lambda|+m}\|_{LE} + \sum_{i=1}^{2} \sup_{R,U} TR^{\frac12}U^{\frac12}\|G_i^{\le|\Lambda|+m}\|_{L^2(C_T^{R,U})} + T^{\frac32}\|r\bar{G}_i^{\le|\Lambda|+m}\|_{LE^*(C_T)}.$$

Using this in (7.14) and (7.12) improves the above estimate to

$$|F^{\Lambda}_{\alpha\beta}| \lesssim C_2 \frac{\log\langle t-r\rangle}{\langle t\rangle\langle t-r\rangle}, \qquad |\nabla F^{\Lambda}_{\alpha\beta}| \lesssim C_2 \frac{\log\langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^2}.$$

One then again uses the pointwise estimates above in the one dimensional reduction to improve the pointwise bounds near the cone, followed by improving the bound inside through (7.14) and derivative bounds near the cone through (7.12); see [23] for more details. After two more iterations, (7.1) follows.

7.5. **Peeling estimates.** Our starting point here is the bound (7.1), which was largely obtained using the wave equation solved by F. However, while bounds of the form (7.1) are optimal for the scalar wave equation, they are not optimal in the case of the Maxwell tensor. Heuristically, this is due to the fact that at the leading order the electromagnetic tensor F is actually the derivative of a potential, and derivatives of the solution decay better away from the light cone even in the case of a scalar wave equation.

In order to improve our estimates, we first note that (7.1) holds for the tensor components evaluated in the null frame. We shall use the standard notation

$$\phi_{-,A} = F_{uA}, \qquad \phi_0 = \frac12(F_{uv} + iF_{AB}), \qquad \phi_{+,A} = F_{vA}.$$

We first note that $\partial_v F_{\alpha\beta}$ satisfies a better decay bound than (7.1) near the cone. Indeed, since

$$\partial_v = \frac1t S + \frac{t-r}{t}\partial_r$$

by (7.1) we have

$$|\partial_v F^{\Lambda}| \lesssim \kappa_1 \frac{1}{\langle r\rangle\langle t\rangle\langle t-r\rangle^2}.$$

It is also immediate that since $e_{A,B} \approx \frac1r\Omega$ we have

$$|e_{A,B}F^{\Lambda}| \lesssim \kappa_1 \frac{1}{\langle r\rangle\langle t\rangle\langle t-r\rangle^2}.$$

We would now like to improve the $\partial_u F$ term. By using the Maxwell system, in particular

$$\nabla^{\alpha}F_{\alpha u} \in S^Z(1)G_2, \qquad \nabla_{[u}F_{AB]} \in S^Z(1)G_1, \qquad \nabla_{[u}F_{vA]} \in S^Z(1)G_1$$

and (7.1) one obtains improved bounds for $\partial_u \phi_0^\Lambda$ and $\partial_u F_{vA}^\Lambda$:

$$|\partial_u \phi_0^\Lambda|, |\partial_u F_{vA}^\Lambda| \lesssim \kappa_1 \frac{1}{\langle r\rangle\langle t\rangle\langle t-r\rangle^2}.$$

After integration on constant v slices, one can improve the pointwise bounds near the cone to

$$|\phi_0^\Lambda|, |F_{vA}^\Lambda| \lesssim \kappa_1 \frac{1}{\langle t\rangle^2\langle t-r\rangle}. \tag{7.21}$$

In order to continue, let $\hat{\phi}_0 := (*F)_{AB} + iF_{AB}$. We will derive the wave equation that $r\hat{\phi}_0$ satisfies.

Lemma 7.2. *The middle component $\hat{\phi}_0$ satisfies*

$$\begin{aligned}\Box(r\hat{\phi}_0{}^{\Lambda}) \in& S^Z(r)(G_1^{\leq|\Lambda|+m}, G_2^{\leq|\Lambda|+m}) + \partial(S^Z(r^{-1})\hat{\phi}_0{}^{\leq|\Lambda|+m})+\\ &S^Z(r^{-2})(\hat{\phi}_0{}^{\leq|\Lambda|+m}) + \partial(S^Z(r^{-2})F^{\leq|\Lambda|+m}) + S^Z(r^{-3})(F^{\leq|\Lambda|+m}).\end{aligned} \tag{7.22}$$

Proof. For spherically symmetric metrics of the form $h_{ab}dx^a dx^b + r^2 d\omega^2$, the lemma has been proved in Section 4 of [28]. We will adapt their proof for our more general case.

Let us consider the metric (recall (2.2))

$$\hat{g} = r^{-2}(1+g_\omega(r))^{-1}g = \tilde{g} + r^{-2}(1+g_\omega(r))^{-1}g_{sr}, \quad \tilde{g} := r^{-2}(1+g_\omega(r))^{-1}(-dt^2+dr^2)+d\omega^2.$$

We note that in Cartesian coordinates the the metric coefficients of $\hat{g}$ and $\tilde{g}$ and their corresponding Christoffel symbols and curvature tensor satisfy the relations

$$\hat{g}_{\alpha\beta} \in S^Z(r^{-2}), \quad \hat{\Gamma}^\gamma_{\alpha\beta} \in S^Z(r^{-1}), \quad \hat{R}^\delta_{\alpha\beta\gamma} \in S^Z(r^{-2}) \tag{7.23}$$

$$\hat{g}_{\alpha\beta} - \tilde{g}_{\alpha\beta} \in S^Z(r^{-4}), \quad \hat{\Gamma}^\gamma_{\alpha\beta} - \tilde{\Gamma}^\gamma_{\alpha\beta} \in S^Z(r^{-3}), \quad \hat{R}^\delta_{\alpha\beta\gamma} - \tilde{R}^\delta_{\alpha\beta\gamma} \in S^Z(r^{-4}). \tag{7.24}$$

We will also use the spherical coordinates (t, r, ϕ, θ) so that locally $\{e_A, e_B\} = \{\frac{1}{r}\partial_\phi, \frac{1}{r\sin\theta}\partial_\theta\}$. Since the Maxwell system is conformally invariant, we have due to (7.3) that

$$\Box_{\hat{g}} F_{CD} - \hat{R}_C{}^\gamma F_{\gamma D} - \hat{R}_D{}^\gamma F_{C\gamma} + \hat{R}_{CD}{}^{\gamma\delta}F_{\gamma\delta} \in S^Z(r^4)(G_1^{\leq 1}, G_2^{\leq 1}), \qquad C, D \in \{\phi, \theta\}.$$

Moreover the metric $\tilde{g}$ is of the form described in Section 4 of [28], so it is easy to check that

$$-\tilde{R}_C{}^\gamma F_{\gamma D} - \tilde{R}_D{}^\gamma F_{C\gamma} + \tilde{R}_{CD}{}^{\gamma\delta}F_{\gamma\delta} = 0,$$

which implies, due to (7.24) that

$$(\hat{R}_C{}^\gamma - \tilde{R}_C{}^\gamma)F_{\gamma D}, \quad (\hat{R}_C{}^\gamma - \tilde{R}_D{}^\gamma)F_{C\gamma}, \quad (\hat{R}_{CD}{}^{\gamma\delta} - \tilde{R}_{CD}{}^{\gamma\delta})F_{\gamma\delta} \in S^Z(1)F.$$

We thus obtain

$$\Box_{\hat{g}} F_{CD} \in S^Z(r^4)(G_1^{\leq 1}, G_2^{\leq 1}) + S^Z(1)F. \tag{7.25}$$

We can now trace with respect to the the volume form ϵ^{CD} of $\mathbb{S}^2$. We obtain, with α signifying Cartesian coordinates:

$$\Box_{\hat{g}}(\epsilon^{CD}F_{CD}) - \epsilon^{CD}\Box_{\hat{g}}F_{CD} = 2\hat{\nabla}_\alpha((\hat{\nabla}^\alpha\epsilon^{CD})F_{CD}) - (\hat{\nabla}_\alpha\hat{\nabla}^\alpha\epsilon^{CD})F_{CD}.$$

Since for the spherically symmetric metric $\tilde{g}$ we know that $\tilde{\nabla}\epsilon = 0$, we obtain that

$$\hat{\nabla}^\alpha\epsilon^{CD} \in S^Z(r^{-1})$$

and thus, taking into account that $F_{CD} \in S^Z(r^2)F$,

$$\Box_{\hat{g}}(\epsilon^{CD} F_{CD}) - \epsilon^{CD}\Box_{\hat{g}} F_{CD} \in \partial(S^Z(r)F) + S^Z(1)F. \tag{7.26}$$

Finally, (7.25), (7.26) and the fact that $\epsilon^{CD}F_{CD} = r^2 F_{AB}$ imply

$$\Box_{\hat{g}}[r^2 F_{AB}] \in S^Z(r^4)(G_1^{\leq 1}, G_2^{\leq 1}) + S^Z(1)F + \partial(S^Z(r)F). \tag{7.27}$$

On the other hand, for any function ψ we have

$$\Box_g \psi - \frac{1}{6}R\psi = r^{-3}\Big(\Box_{\hat{g}}(r\psi) - \frac{1}{6}\hat{R}(r\psi)\Big). \tag{7.28}$$

In particular for $\psi = rF_{AB}$ we obtain, using (7.27),

$$\Box_g[rF_{AB}] \in S^Z(r)(G_1^{\leq 1}, G_2^{\leq 1}) + S^Z(r^{-2})(F_{AB}) + \partial(S^Z(r^{-2})F) + S^Z(r^{-3})(F).$$

One obtains a similar equation for $(*F)_{AB}$. We thus get

$$\Box_g[r\hat{\phi}_0] \in S^Z(r)(G_1^{\leq 1}, G_2^{\leq 1}) + S^Z(r^{-2})(\hat{\phi}_0) + \partial(S^Z(r^{-2})F) + S^Z(r^{-3})(F).$$

We can now replace $\Box_g$ above by the operator $P = \Box + Q$ from (2.3), and the conclusion follows after commuting with elements of Z. $\Box$

We can now apply the following lemma, proved in [23] (Lemma 3.20) for $m = -2, 1$:

Lemma 7.3. *Consider a smooth function f supported in $\{\frac{t}{2} \leq r \leq t\}$ so that*

$$|f| + |Sf| + |\Omega f| + \langle t-r\rangle |\partial_r f| \lesssim \frac{\log^m\langle t-r\rangle}{t^3\langle t-r\rangle}, \quad m \in \mathbb{Z}. \tag{7.29}$$

and h supported in $\{0 < r_e \leq r \leq t\}$ so that

$$|h| \lesssim \frac{\log^m\langle t-r\rangle}{tr^3\langle t-r\rangle}. \tag{7.30}$$

Then the forward solution w to

$$\Box w = \partial_t f + h$$

satisfies the bound

$$|w| \lesssim \frac{\log^{m+2}\langle t-r\rangle}{t\langle t-r\rangle^2}. \tag{7.31}$$

Proof. We write $w = w_1 + w_2$, where

$$\Box w_1 = \partial_t f, \qquad \Box w_2 = h.$$

Let us first bound w_2. We use the one-dimensional reduction (7.20) and decompose D_{tr} dyadically as

$$D_{tr} = \bigcup_{R \leq t} D_{tr}^R, \quad D_{tr}^R = D_{tr} \cap \{R < \rho < 2R\}.$$

For $R < (t-r)/8$, we have that $\rho \approx R$ and $\langle s-\rho\rangle \approx \langle t-r\rangle$ in D_{tr}^R. Thus (7.30) implies that

$$\int_{D_{tr}^R}\int_{\mathbb{S}^2} \rho|h^{\leq n}| d\omega\, ds\, d\rho \lesssim \frac{\log^m\langle t-r\rangle}{\langle t-r\rangle^2}.$$

On the other hand, for $R > (t-r)/8$ we obtain by (7.30) (here $u = s - \rho$):

$$\int_{D_{tr}^R} \int_{\mathbb{S}^2} \rho |h^{\leq n}| d\omega \, ds \, d\rho \lesssim R^{-2} \int_0^{t-r} \frac{\log^m \langle u \rangle}{\langle u \rangle} du \lesssim R^{-2} \log^{m+2} \langle t-r \rangle.$$

Dyadically summing in R shows that

$$|w_2| \lesssim \frac{\log^{m+2} \langle t-r \rangle}{r \langle t-r \rangle^2},$$

and using this bound in (7.14) yields (7.31).

On the other hand, we can write $w_1 = \partial_t v$, where v is the forward solution to $\Box v = f$. We first note that (7.29) also implies that

$$|(t\partial_i + x_i \partial_t) f| \lesssim \frac{\log^m \langle t-r \rangle}{t^n \langle t-r \rangle}.$$

Via the one dimensional reduction as above applied to v, ∇v Ωv, Sv and $(t\partial_i + x_i \partial_t) v$ we obtain

$$|v| + |\nabla v| + |Sv| + |\Omega v| + \sum_i |(t\partial_i + x_i \partial_t) v| \lesssim \frac{\log^{m+2} \langle t-r \rangle}{t \langle t-r \rangle}.$$

The above left hand side dominates $\langle t-r \rangle \partial_t v$; therefore the proof of the lemma is complete. $\Box$

In order to use the lemma, we recall that

$$\partial = \partial_t + S^Z(\frac{1}{r})(S, \Omega).$$

Let χ be a smooth cutoff selecting the region $r \leq t/2$. We can rewrite (7.22) as

$$\Box(r\hat{\phi_0}^{\Lambda}) = S^Z(r)(G_1^{\leq |\Lambda|+m}, G_2^{\leq |\Lambda|+m}) + \partial_t f + h$$

where

$$f = (1-\chi)\Big(S^Z(r^{-1})\hat{\phi_0}^{\leq |\Lambda|+m} + S^Z(r^{-2}) F^{\leq |\Lambda|+m}\Big)$$

$$h = \partial_t \Big(\chi(S^Z(r^{-1})\hat{\phi_0}^{\leq |\Lambda|+m} + S^Z(r^{-2}) F^{\leq |\Lambda|+m})\Big) + S^Z(r^{-2})(\hat{\phi_0}^{\leq |\Lambda|+m}) + S^Z(r^{-3})(F^{\leq |\Lambda|+m})$$

Due to (7.22), (7.21), (7.1), and (7.11) we see that

$$|f| + |Sf| + |\Omega f| + \langle t-r \rangle |\partial_r f| \lesssim \frac{1}{t^3 \langle t-r \rangle}, \quad |h| \lesssim \frac{1}{tr^3 \langle t-r \rangle}.$$

We can now use Lemma 7.3 (with $m = 0$), to improve the bounds on $\hat{\phi_0}$ to

$$|\hat{\phi_0}^{\Lambda}| \lesssim \kappa \frac{\log^2 \langle t-r \rangle}{r \langle t \rangle \langle t-r \rangle^2}. \tag{7.32}$$

We can use the new bound (7.32) to improve the decay of f and h to

$$|f| + |Sf| + |\Omega f| + \langle t-r \rangle |\partial_r f| \lesssim \frac{1}{t^3 \langle t-r \rangle \log^2 \langle t-r \rangle}, \quad |h| \lesssim \frac{1}{tr^3 \langle t-r \rangle \log^2 \langle t-r \rangle}.$$

By applying Lemma 7.3 (with $m = -2$) we can remove the logarithm in (7.32), and obtain the desired estimate for ϕ_0 near the cone:

$$|\phi_0^\Lambda| \lesssim \kappa \frac{1}{r\langle t\rangle\langle t-r\rangle^2}. \tag{7.33}$$

We will later improve in the region $r \le t/2$ by replacing the r factor in the denominator by a t factor.

Note that (7.33) implies, in conjunction with (7.12), that in the region $\{r > t/2\}$ we have

$$|\nabla\phi_0^\Lambda| \lesssim \kappa \frac{1}{r^2\langle t-r\rangle^3}. \tag{7.34}$$

Since ϕ_- is a one-form on the sphere, Hodge theory tells us that

$$\|\phi_-\|_{L^2(\mathbb{S}^2)} \lesssim \|\nabla\!\!\!\!/\,\phi_-\|_{L^2(\mathbb{S}^2)}.$$

On the other hand, part of the Maxwell system gives that

$$\nabla^\alpha F_{\alpha u} \in S^Z(1)G_2, \qquad \nabla_{[u}F_{AB]} \in S^Z(1)G_1 \tag{7.35}$$

which in turn implies that

$$\|\nabla\!\!\!\!/\,\phi_-\|_{L^\infty(|x|=R)} \lesssim \|\partial_u\phi_0\|_{L^\infty(|x|=R)} + \frac{1}{R}\|F\|_{L^\infty(|x|=R)} + \sum_{i=1}^2 \|G_i\|_{L^\infty(|x|=R)}.$$

After taking derivatives in (7.35) and applying Sobolev embeddings on the sphere of radius r, we obtain the desired bound for F_{uA} near the cone:

$$|F_{uA}^\Lambda| \lesssim \kappa \frac{1}{\langle r\rangle\langle t-r\rangle^3}. \tag{7.36}$$

To complete the proof of the peeling estimates near the cone, we note that the Maxwell system, in particular the equations

$$\nabla_{[u}F_{vA]} = (G_1)_{uvA}, \qquad \nabla^\alpha F_{\alpha A} = -\ast(G_2)_A$$

combined with the previous bounds for ϕ_- and ϕ_0 imply that

$$|\partial_u F_{vA}^\Lambda| \lesssim \kappa \frac{1}{r^2\langle t\rangle\langle t-r\rangle^2}.$$

After integration on constant v slices, we obtain the desired bound for F_{vA} near the cone:

$$|F_{vA}^\Lambda| \lesssim \kappa \frac{1}{r\langle t\rangle\langle t-r\rangle^2}. \tag{7.37}$$

Finally, we note that (7.33), (7.36) and (7.37) imply that $|F^{\le n}| \lesssim \kappa \frac{1}{\langle r\rangle\langle t-r\rangle^3}$. Taking into account (7.14), we can replace the r factor in the denominator with a t factor and conclude the proof. One can also easily see that (3.2) follows from the proof of Lemma 6.1 by using the decay of F just proved.

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