A Functional Analyst Looks at Concentration Compactness

This is a purely expository talk inspired by the following problem: suppose that $\{u_n\}$ is a bounded sequence in $L^2(\mathbb{R}^d)$. Owing to the action of translations and dilations on $\mathbb{R}^d$, such a sequence need not have any convergent subsequence, as the examples $u_n(x) = u(x+n)$, $u_n(x) = n^{1/2}u(nx)$, and $u_n(x) = n^{-1/2}u(x/n)$ in $L^2(\mathbb{R})$ show. Is there a nonetheless a convergent subsequence, modulo the action of translations and dilations?

The answer to the question is yes in the sense that, by passing to a subsequence of $\{u_n\}$, one can write u_n as a sum of at most countably many translated and dilated *profiles*, that is, model L^2 functions that describe the asymptotic behavior of the sequence (in the above examples, there is only one profile!). Profile decompositions arise in the calculus of variations, in the study of harmonic maps, in the analysis of semilinear elliptic equations and, believe it or not, in scattering theory.

In this talk we will discuss an illuminating 'toy model' of profile decomposition due to Terence Tao,[1] and then put the question in a functional-analytic framework pioneered by Kyril Tintarev and exposed in a beautiful monograph of Tintarev and Fieseler, *Concentration Compactness: Functional Analytic Grounds and Applications*. We will outline the proof of Tintarev-Fieseler's main result, the existence and uniqueness of profile decompositions in suitable function spaces.

[1] See https://terrytao.wordpress.com/2008/11/05/concentration-compactness-and-the-profile-decomposition/