Homework 1 - Solutions

1. Prove the following statement by contradiction.

   The sum of two even numbers is always even.

   Let m and n be two even numbers and assume for contradiction that $m+n$ is not even. If $m+n$ is not even, then there exists $k \in \mathbb{N}$ such that $m+n = 2k+1$. Solve for m to get $m = 2k+1-n$. Since n is even, there exist $r \in \mathbb{N}$ such that $n = 2r$. But then $m = 2k+1-2r$ which is an odd number contradicting m being even. Therefore, $m+n$ is even.

2. Prove the statement using the ϵ, δ definition of the limit.

   $$\lim_{x \to 2} x^2 - 5x + 3 = -3$$

   Let $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{2}$. Then if $|x-2| < \delta$, we have

   $$|f(x) - L| = |x^2 - 5x + 3 - (-3)| = |x^2 - 5x + 6| = |x-2||x-3| < \delta|x-3|$$

   Now, $|x-2| < \delta$. Specifically, we can assume $|x-2| < 1$. This means that $|x-3| < 2$. So we have

   $$|f(x) - L| = |x-2||x-3| < 2\delta = \epsilon$$

3. Prove the statement using the ϵ, M definition of the limit.

   $$\lim_{x \to \infty} \frac{x-1}{2x} = \frac{1}{2}$$

   Let $\epsilon > 0$ be given. Let $M = \frac{2}{\epsilon}$. Then for $x > M$ we have

   $$|f(x) - L| = \left| \frac{x-1}{2x} - \frac{1}{2} \right| = \left| \frac{-1}{2x} \right| < \left| \frac{1}{2M} \right| = \epsilon$$

4. Evaluate

$$\lim_{x\to\infty} \frac{12x - 13x^2 + 5x^4}{2x^4 - 3x + 1}$$

One way to rigorously approach evaluating this limit is to multiply the numerator and denominator by $\frac{1}{x^4}$. This gives

$$\lim_{x\to\infty} \frac{12x - 13x^2 + 5x^4}{2x^4 - 3x + 1} = \lim_{x\to\infty} \frac{\frac{12}{x^3} - \frac{13}{x^2} + 5}{2 - \frac{3}{x^3} + \frac{1}{x^4}}$$

As x approaches infinity, all of the terms with x in the denominator go to zero. Using a combination of limit properties, we get

$$\lim_{x\to\infty} \frac{\frac{12}{x^3} - \frac{13}{x^2} + 5}{2 - \frac{3}{x^3} + \frac{1}{x^4}} = \lim_{x\to\infty} \frac{5}{2} = \frac{5}{2}$$

5. Evaluate

$$\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$$

Here we use the squeeze theorem. For all $x \neq 0$, $|\sin(\frac{1}{x})| \leq 1$. This means

$$\lim_{x\to 0} x^2 e^{-1} \leq \lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} \leq \lim_{x\to 0} x^2 e^{1}$$

As x approaches zero, the left and rightmost limits go to zero. By the squeeze theorem,

$$\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$$