Homework 2 - Due 10:00 AM on Wednesday August 7

Solutions should be clear and organized. Make sure you justify your work.

1. Find the partial derivatives of $f(x, y, z) = x^2y^3z^4\cos(x)$

$$f_x(x, y, z) = y^3z^4(2x\cos(x) - x^2\sin(x))$$
$$f_y(x, y, z) = 3x^2y^2z^4\cos(x)$$
$$f_z(x, y, z) = 4x^2y^3z^3\cos(x)$$

2. Evaluate

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$

This limit meets the requirement of L'Hopitals rule. Therefore,

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$
$$= \lim_{x\to 0} \frac{e^x - 1}{2x}$$
$$= \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

3. Let $F(x) = \int_2^{x^2} t^2 + 1\, dt$, find $F'(x)$. (Hint: Use the fundamental theorem of calculus)

Using the chain rule and the fundamental theorem of calculus gives

$$F'(x) = (x^2 + 1)(2x)$$

4. Evaluate $\int \frac{x}{x^2+1}\, dx$

Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$. We can rewrite the integral as

$$\int \frac{x}{x^2 + 1}\, dx = \int \frac{x}{u}\frac{du}{2x} = \int \frac{1}{2u}\, du = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln(x^2 + 1)$$

5. Evaluate $\int e^x \sin(x)\, dx$

Use integration by parts letting $u = e^x$ and $dv = \sin(x)\, dx$ to get

$$\int e^x \sin(x)\, dx = -e^x\cos(x) + \int e^x\cos(x)\, dx$$

Use integration by parts again letting $u = e^x$ and $dv = \cos(x)\, dx$ to get

$$\int e^x \sin(x)\, dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\, dx$$

Now move all the integrals to the left side to get

$$2\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) + C$$

Equivalently,

$$\int e^x \sin(x)\,dx = \frac{1}{2}\left(-e^x\cos(x) + e^x\sin(x)\right) + C$$

6. Evaluate $\int \frac{3x+11}{x^2-x-6}\,dx$

Apply partial fraction decomposition to get

$$\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

Equivalently,

$$3x + 11 = A(x+2) + B(x-3)$$

Let $x = -2$ to get $5 = B(-5) \implies B = -1$.

Let $x = 3$ to get $20 = 5A \implies A = 4$. Therefore,

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx = 4\ln|x-3| - \ln|x+2| + C$$

7. Suppose that f is a function with the property that $|f(x)| \le x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

Proof: Since f has the property $|f(x)| \le x^2$ for all x, it follows that $0 \le |f(0)| \le 0^2 = 0$, hence $f(0) = 0$. We also have that

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0}\frac{f(h)}{h} \end{aligned}$$

From the defining property, we have $-x^2 \le f(x) \le x^2$, and hence $-|x| \le \frac{f(x)}{x} \le |x|$. We know that

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0.$$

By the Squeeze Theorem, $\lim_{x\to 0}\frac{f(x)}{x}$ exists and is equal to 0. Thus, $f'(0) = 0$.